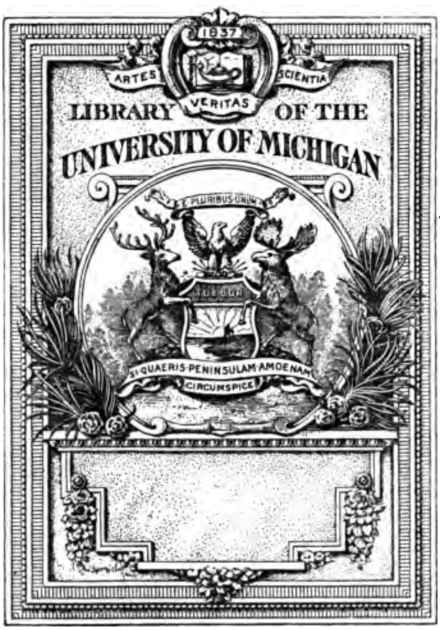
ALTERNATING CURRENT

COMMUTATOR MOTORS

BY A. S. M'ALLISTER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ALTERNATING

CURRENT COMMUTATOR MOTORS

A thesis presented to the University Faculty of Cornell University for the Degree of Doctor of Philosophy.

BY

A. S. MᶜALLISTER

JUNE, 1905

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ALTERNATING CURRENT COMMUTATOR MOTORS. REPULSION MOTOR.*

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In dealing with the phenomena connected with the operation of alternating current motors of the commutator type, it must be constantly borne in mind that the machine possesses simultaneously the electrical characteristics of both a direct current motor and a stationary alternating current transformer. The statement just made must not be confused with a somewhat similar one which is applicable to polyphase induction motors, since only with regard to its mechanical characteristics does an inducduction motor resemble a shunt-wound direct current machine, its electrical characteristics being equivalent in all respects to those of a stationary transformer.

Before discussing the performance of repulsion motors, it is well to investigate a few of the properties common to all commutator type, alternating current machines. It will be recalled that when current flows through the armature of a direct current machine, magnetism is produced by the ampere turns of the armature current, such magnetism tending to distort the flux from the field poles. In the familiar representation of the magnetic circuit of machines,—the two pole model,—the armature magnetism is at right angles to the field magnetism, the armature current producing magnetic poles in line with the brushes. The amount of this magnetism depends directly on the value of the armature current and the permeability of the magnetic path. When alternating current is used, the change of the magnetism with the periodic change in the current produces an alternating e.m.f. which being proportional to the rate of change of the magnetism will be in time-quadrature to the current. The armature winding thus acts in all respects similarly to an induction coil.

It is not essential that the current to produce the alternating flux flow through the armature coils in order that the alternating e.m.f. be developed at the commutator. Under whatsoever conditions the armature conductors be subject to changing flux a corresponding e.m.f. will be generated, in mechanical line

* **Abstract of thesis for Ph.D. degree, Cornell University.**

with the flux and in time-quadrature to it. Referring to Fig. 1 which represents a direct current armature situated in an alternating field, having two pair of brushes, one in mechanical line with the alternating flux and one in mechanical quadrature thereto. When the armature is stationary an e.m.f. will be generated at the brushes *A* and *A* due to the transformer action of the flux, but no measurable e,m.f. will exist between *B* and *B*. As seen above, this e.m.f. is in time-quadrature with the field (transformer) flux and as will be seen later, its value is unaltered by any motion of the armature. At any speed of the armature, there will be generated at the brushes *B* and *B* an e.m.f. proportional to the speed and to the field magnetism and in time-phase with the magnetism. At a certain speed this "dynamo" e.m.f. will be equal in effective value to the "transformer" e.m.f. at *A* and *A*, though it will be in time-quadrature to it. This critical speed will hereafter be referred to as the "synchronous" speed, and with the two-pole model shown in Fig. 1, it is characterized by the fact that in whatsoever posi-

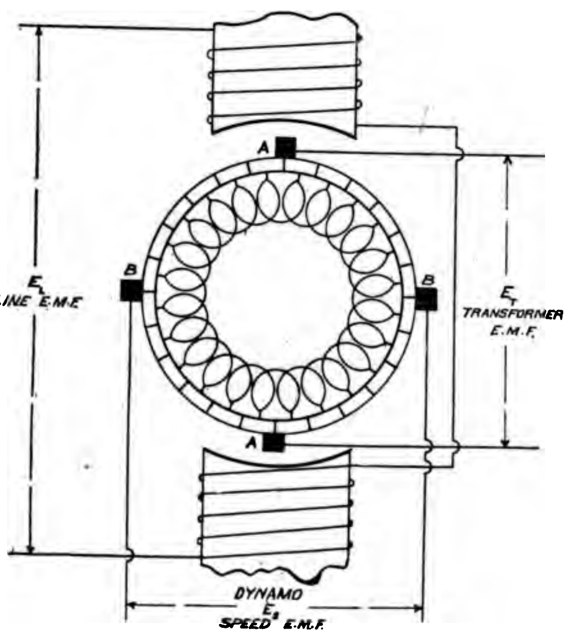


FIG. 1 — ELECTROMOTIVE FORCES PRODUCED IN AN ALTERNATING FIELD.

tion on the armature a pair of brushes be placed across a diameter, the e.m.f. between the two brushes will be the same and will have a relative time-phase position corresponding to the mechanical position of the brushes on the commutator.

A little consideration will show that the individual coils in which the maximum e.m.f. is generated by transformer action are situated upon the armature core under brushes B or B, although the difference of potential between the brushes B and B is at all times of zero value as concerns the transformer action. A similar study leads to the conclusion that the e.m.f. generated by dynamo speed action appears as a maximum for a single coil when the coil is under brush A or A. Assuming as zero position, the place under brush A and that at synchronous speed the e.m.f. generated in a coil at this position is e. Then the e.m.f. in a coil at b will equal e also. A coil α degrees from this position will have generated in it a speed e.m.f. of $e \cos \alpha$ and a transformer e.m.f. of $e \cos (\alpha \pm 90) = \mp e \sin \alpha$. Since these two component e.m.f.s are in time quadrature the resultant will be $V = \sqrt{(e \cos \alpha)^2 + (\pm e \sin \alpha)^2} = e$ and is the same for all values of α. The time-phase position of the resultant, however, will vary directly with α or with the mechanical position of the coil. From these facts it is seen that at synchronous speed the effective value of the e.m.f. generated per coil at all positions is the same and that there is no neutral e.m.f. position on the commutator.

In a repulsion motor as commercially constructed, the secondary consists of a direct current armature upon the commutator of which brushes are placed in positions 180 electrical degrees apart and directly short circuted upon themselves, as shown in the two-pole model of Fig. 2. The stationary primary member consists of a ring core containing slots more or less uniformly spaced around the air-gap. In these slots are placed coils so connected that when current flows in them definite magnetic poles will be produced upon the field core. The brushes on the commutator are given a location some 15 degrees from the line of polarization of the primary magnetism, or more properly expressed, the brushes are placed about 15 degrees from the true transformer position. That component of the magnetism which is in line with the brushesproduces current in the secondary by transformer action, and this current gives a torque to the rotor due to the presence of the other component of magnetism in mechanical quadrature to the secondary current.

It is possible to make certain assumptions as to the relative values of the magnetism in mechanical line with, and in mechanical quadrature to the brush line and thus to derive the fundamental equations of the machine. It is believed, however, that the facts can be more clearly presented and the treatment simplified without sacrifice of accuracy if the assumption be made that the primary coil is wound in two parts, one in mechanical line and the other in mechanical quadrature with the axial brush position as shown in Fig. 2. It will be noted that the two fields produced by the sections of the primary coil if

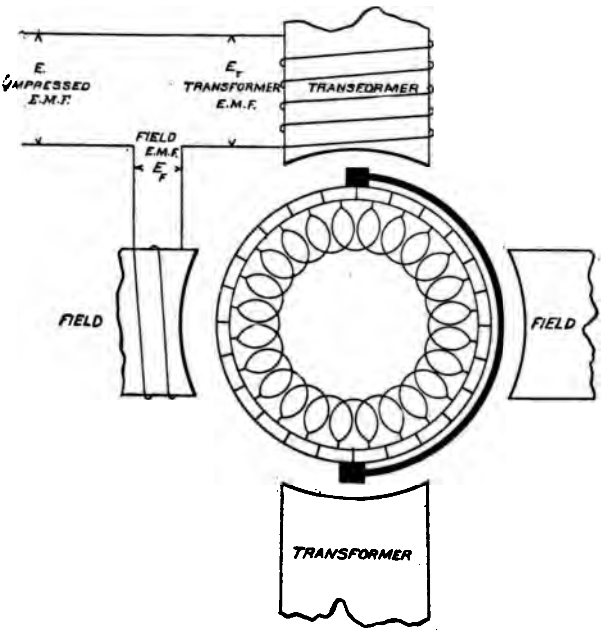


FIG. 2.—TWO-POLE MODEL OF IDEAL REPULSION MOTOR.

there were no disturbing influence present, would have a resultant position relative to the brush line depending upon the ratio of the strengths of the two magnetisms. The angle which the resultant field would assume can be represented by β having a value such that cotan $\beta = \frac{\phi_t}{\phi_f}$ where ϕ_t is the flux through transformer coil and ϕ_f is flux through field coil. If n be the ratio of turns on the transformer poles to those on the field poles, then for any value of current in these coils (no secondary current)

$$\frac{\phi_t}{\phi_f} = n \text{ or } n = \text{cotan } \beta \qquad (1)$$

It is understood that in Fig. 2, the core material is considered to be continuous and that in the two-pole model represented both field poles and both transformer poles are supposed to be properly wound.

In Fig. 2, let it be assumed that the machine is stationary and that a certain e.m.f., E, is impressed upon the primary circuits, the secondary being on short circuit. The flux which the primary current tends to produce in the transformer pole produces by its rate of change an e.m,f. in the secondary, and this e.m.f. causes opposing current to flow in the closed secondary circuit. If the transformer action is perfect and the transformer coil and armature circuits are without resistance and local leakage reactance, then the magnetomotive force of the armature current equals that of the current in the transformer coil, and the resultant impedance effect of the two circuits is of zero value, so that the full primary e.m.f., E, is impressed upon the field coil, that is to say, with armature stationary $E_t = O$, and $E_f = E$.

It remains now to investigate the effect of speed on the electromotive forces of the transformer and field coils. Assume a certain flux Φ_f in the field coil. At speed S the armature conductors will cut this flux and at each instant there will be generated an e.m.f. therein proportional to $S\phi_f$, and therefore, in time-phase with the flux. This e.m.f. would tend to cause current to flow in the closed armature circuit, which current would produce magnetism in line with the brushes, and, since the armature circuit has zero impedance, (assumed) the flux so produced will be of a value such that its rate of change through the armature coils just equals the e.m.f. generated therein by speed action. At synchronous speed, the secondary being closed, the flux in line with the brushes must equal that in line with the field poles, since the e.m.f. generated by the rate of change of the flux in the direction of the brushes must equal that generated at the brushes due to cutting the field magnetism, and at a speed which has been termed synchronous these two fluxes are equal, as previously discussed. At this speed the two fluxes are equal but they are in time-quadrature one to the other. At other speeds the two fluxes retain the quadrature time-phase position, but the ratio of the effective values of the two fluxes varies directly with the speed.

Giving to synchronous speed a value of unity, at any speed,

S, the transformer flux may be expressed by the equation

$$\phi_t = S\phi_f \qquad (2)$$

effective values being used throughout. Letting ϕ be the maximum value of the field flux and reckoning time in electrical degrees from the instant when the field flux is maximum, at any time γ, the instantaneous field flux is

$$\phi_f = \phi \cos \gamma \qquad (3)$$

and the transformer flux is

$$\phi_t = S\phi \sin \gamma \qquad (4)$$

These are the fundamental magnetic equations of the ideal repulsion motor.

If at a certain speed S, the effective value of e.m.f. across the field coil be F, requiring an effective flux of ϕ_f, then across the transformer coil there will be an effective e.m.f. of

$$T = nSF \qquad (5)$$

due to the flux $S\phi_f$. Since the fluxes are in time-quadrature, the e.mf.s are likewise in time quadrature, so that the impressed e.m.f. E must have a value such that

$$E = \sqrt{F^2 + T^2} \qquad (6)$$

This is the fundamental electromotive force equation of the repulsion motor.

The current which flows through the field coil is

$$I = \frac{F}{X} \qquad (7)$$

where X is the inductive reactance of the field coil. Equation (7) gives the value of the primary circuit current and is the fundamental primary current equation.

The secondary armature current in general consists of two components, that equal in magnetomotive force and opposite in phase to the primary transformer current, and that necessary to produce the flux in line with the brushes. With a ratio of effective armature turns to field turns of a, the opposing transformer current is

$$I_t = \frac{nI}{a} \qquad (8)$$

and the current which produces the transformer poles is

$$I_f = \frac{SI}{a} \qquad (9)$$

These component currents are in time-quadrature, so that the resultant secondary current is

$$I_s = \sqrt{I_t^2 + I_f^2} \qquad (10)$$

This is the fundamental equation for the secondary current. Combining (8) (9) and (10)

$$I_s = \frac{I}{a}\sqrt{n^2 + S^2} \qquad (11)$$

It has been seen that the e.m.f. T is in time-quadrature to the field circuit e.m.f., F. Now the current is in time-quadrature with F, and hence, is in time-phase with T. Therefore, of the total primary e.m.f. E, the part T is in phase with the current, from which fact it is seen that the power factor is

$$\cos\theta = \frac{T}{E} \qquad (12)$$

Power,

$$P = EI\cos\theta = \frac{EFT}{XE} = IT \qquad (13)$$

Torque,

$$D = \frac{P}{S} = \frac{IT}{S} = \frac{ISnF}{S} = InF$$

$$D = InF = InXI = I^2nX \qquad (14)$$

$$E^2 = F^2 + T^2 = F^2\,(1 + S^2n^2) \qquad (15)$$

$$F = \frac{E}{\sqrt{1 + S^2n^2}} \qquad (16)$$

$$I = \frac{E}{X\sqrt{1 + S^2n^2}} \qquad (17)$$

$$I_s = \frac{E\sqrt{n^2 + S^2}}{aX\sqrt{1 + S^2n^2}} \qquad (18)$$

when $n = 1$, that is at $\beta = 45°$ see (1)

$I_s = \dfrac{E}{aX}$ and is *constant* at all speeds.

when $S = 1$, that is at synchronism for *any* value of n.

$I_s = \dfrac{E}{aX}$ which is seen to be equal to the primary current at starting, (when $a = 1$)

when $S = 1$ the secondary current

$$I_s = \frac{I}{a}\sqrt{n^2 + 1}$$

and leads the primary current by angle $\cot^{-1} n = \beta$ or angle of brush shift. See equation (1).

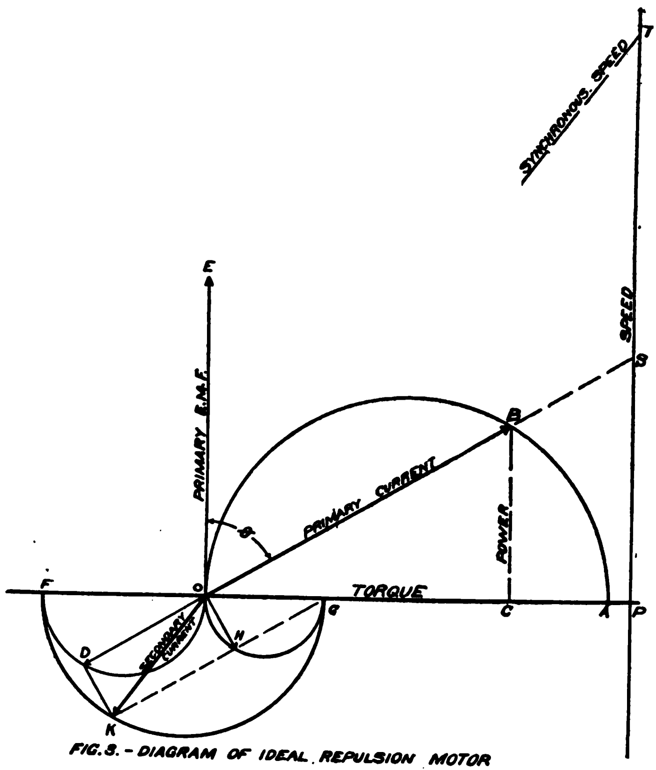


FIG. 3. - DIAGRAM OF IDEAL REPULSION MOTOR

The above equations can be expressed graphically by a simple diagram as shown in Fig. 3. The diagram is constructed as follows: *OE* is the constant line e.m.f. *OA* at rt. angles to *OE* is the line current at starting, *OBA* is a semicircle, *OF* in phase opposition to *OA* is the secondary current at starting. *ODF* is a semicircle. *OG*, in phase with *OA*, is the secondary current at infinite speed. *OHG* is a semicircle. It will be noted that the ratio *OA* to *OG* is $na : 1$ and ratio of *OA* to *OF* is $a : n$.

Distances measured from *P* in the direction of *T* represent speed.

The characteristics of the machine may be found at once from Fig. 3. Assuming any speed as *PS*, draw *OS* intersecting the circle *OBA* at *B*. From point *G* draw line *GK* parallel to *OS*. Join *O* and *K*.

OK is secondary current ;
OB is primary current ;
EOS is primary angle of lag ;
BC is power component of primary current ;
BC is power (to proper scale) ;
OC is torque (to proper scale) ;
DOK is angle of lead of secondary current.

At synchronous speed ($S = 1$), cotan $\theta = n$, hence scale of speed can readily be located.

$OD = I_t$, see equation (7).

$OH = I_f$, see equation (8).

The proof of the construction of diagram of Fig. 3 is as follows :

$$\cos\theta = \frac{T}{E} \qquad \text{Eq. (11)}$$

$$E^2 = T^2 + F^2 \qquad \text{Eq. (6)}$$

$$T = SnF \qquad \text{Eq. (5)}$$

$$E^2 = T^2\left(1 + \frac{1}{S^2n^2}\right) \qquad (19)$$

$$E = \frac{T}{Sn}\sqrt{1 + S^2n^2} \qquad (20)$$

$$\cos\theta = \frac{T}{E} = \frac{Sn}{\sqrt{1 + S^2n^2}} \qquad (21)$$

Power component of primary current

$$I_p = I\cos\theta = \frac{E}{X}\frac{Sn}{(1 + S^2n^2)} \qquad (22)$$

$$\text{Sin}\,\theta = \sqrt{1 - \frac{S^2n^2}{1 + S^2n^2}} = \frac{1}{\sqrt{1 + S^2n^2}} \qquad (23)$$

Quadrature component of primary current

$$I_q = I\sin\theta = \frac{E}{X\sqrt{1 + S^2n^2}}\frac{1}{\sqrt{1 + S^2n^2}}$$

$$I_q = \frac{E}{X(1 + S^2n^2)} \qquad (24)$$

$$\frac{I_p}{I_q} = \text{cotan}\,\theta = \frac{E}{X}\frac{sn}{1 + S^2n^2}\frac{X(1 + S^2n^2)}{E}$$

$$\text{cotan}\,\theta = Sn \qquad (25)$$

The cotangent of the angle of lag is directly proportional to the speed, the proportionality constant being the ratio of transformer to field turns.

$$D = \frac{IE\cos\theta}{S} = \frac{E^2n}{X(1 + S^2n^2)} = nEI_q \qquad (26)$$

Torque is proportional to quadrature component of the primary (for given e.m.f.) the proportionality constant being the ratio of transformer to field turns.

$$D = \frac{E^2n}{X(1 + S^2n^2)} = I^2nX \qquad (27)$$

Torque varies as the square of the primary current and in this respect is independent of the speed or the e.m.f.

A comparison of equations (26) and (27) reveals an interesting property of a circle. In Fig. 3 assuming the diameter A O to be unity, $O\,C$ at all valnes of angle θ equals the square of $O\,B$.

From equation (27) it is seen that the torque is at all times positive, even when S is negative. Hence machine acts as generator at negative speed. For the determination of the generator characteristics it is necessary to construct the semicircle omitted in each case in Fig. 3.

It is interesting to observe that the construction of the diagram of Fig. 3 can be completed at once when points F, O, G and A and E are located. Thus the complete performance of the ideal repulsion motor can be determined when E, X, n and a are known. In the construction for ascertaining the value of the secondary current, it will be seen that $O\,K$ is equal to the vector sum of $O\,D$ and $O\,H$, giving the vector $O\,K$. From the properties of vector co-ordinates it will be noted that the point K is located on the semicircle $F\,K\,G$ whose center lies in the line $F\,O\,G$. Therefore if G and F be located, the inner circles $F\,D\,O$ and $O\,H\,G$ need not be drawn, since the point K can be found as the intersection of the line drawn parallel to $O\,B$ from G with the circular arc $F\,K\,G$.

It is to be carefully noted that the above discussion refers to ideal conditions which can never be realized. The circuits have been considered free from resistance and leakage reactance while all iron losses, friction, and brush short circuiting effects have been neglected. The resistance and leakage reactance effects can quite easily be taken into account, but the remaining disturbing influences are subject to considerable error in approximating their values, due primarily to the difficulty in assigning to iron any constant in connection with its magnetic phenomena. It is to be regretted that the so-called complete equations for expressing the characteristics of this type of machinery with al-

most no exception neglect these disturbing influences, and yet these same equations are given forth by the various writers as though they represented the true conditions of operation.

In the ideal motor the apparent impedance is

$$\underline{Z} = \frac{E}{I} = X\sqrt{1 + S^2 n^2} \tag{28}$$

apparent resistance is

$$\underline{R} = Z \cos\theta = XSn \tag{29}$$

since

$$\cos\theta = \frac{T}{E}; \quad T = SnF; \text{ and } E = \sqrt{T^2 + F^2}$$

hence

$$E = \frac{T}{Sn}\sqrt{1 + S^2 n^2}; \quad \cos\theta = \frac{Sn}{\sqrt{1 + S^2 n^2}}$$

apparent reactance is

$$\underline{X} = Z \sin\theta = X \tag{30}$$

since

$$\sin\theta = \sqrt{1 - \frac{S^2 n^2}{1 + S^2 n^2}} = \frac{1}{\sqrt{1 + S^2 n^2}}$$

Let R_f = resistance of field coil
R_t = resistance of transformer coil
R_a = resistance of armature coil
X_a = reactance of armature coil
X_t = reactance of transformer coil
X_f = reactance of field coil

then copper loss of motor circuits will be

$$I^2 (R_f + R_t) + I_a^2 R_a \tag{31}$$

$$I_a = \frac{E}{aX}\frac{\sqrt{n^2 + S^2}}{\sqrt{1 + S^2 n^2}} \tag{18}$$

$$I = \frac{E}{X\sqrt{1 + S^2 n^2}} \tag{17}$$

hence

$$I_a = \frac{I\sqrt{n^2 + S^2}}{a} \tag{32}$$

and copper loss will be

$$I^2\left[(R_f + R_t) + \left(\frac{n^2 + S^2}{a}\right) R_a\right] = I^2 R_m \qquad (33)$$

where R_m is the effective equivalent value of the motor circuit resistance, that is

$$R_m = R_f + R_t + \left(\frac{n^2 + S^2}{a}\right) R_a \qquad (34)$$

Similarly it may be shown that the effective equivalent value of the leakage reactance of the motor circuits is

$$X_m = X_f + X_t + \left(\frac{n^2 + S^2}{a}\right) X_a \qquad (35)$$

If these valves be added to the apparent resistance and reactance of the ideal motor the corresponding effects will be represented in the resultant equations thus

$$\underline{R} = XSn + R_f + R_t + \left(\frac{n^2 + S^2}{a}\right) R_a \qquad (36)$$

and

$$\underline{X} = X + X_f + X_t \left(\frac{n^2 + S^2}{a}\right) X_a \qquad (37)$$

$$Z = \sqrt{\underline{R}^2 + \underline{X}^2} \text{ from (36) and (37)} \qquad (38)$$

$$\cos\theta = \frac{R}{\underline{Z}}\,;\; I = \frac{E}{\underline{Z}} \qquad (39)$$

$$\text{Input} = EI\cos\theta \qquad (40)$$

$$\text{output} = EI\cos\theta - I^2\underline{R_m} = P \qquad (41)$$

$$\text{torque} = \frac{P}{S} = \text{D, etc.} \qquad (42)$$

It will be noted that the short circuiting by the brush of a coil in which an active e.m.f. is generated has thus far not been considered. Referring to Fig. 2, it will be seen that at any speed S there will be generated in the coil under the brush by dynamo speed action an e.m.f.

$$E_s = K\phi_t S \qquad (43)$$

where K is constant. This e.m.f. is in time-phase with the flux ϕ_t. In this coil there will also be generated an e.m.f. by the transformer action of the field flux, such that,

$$E_f = K\phi_f \qquad (44)$$

This e.m.f. is in time-quadrature to ϕ_f. Since ϕ_f and ϕ_t are in time quadrature the component e.m.f.s acting in the coil under

the brush are in time-phase (opposition) so that the resultant e.m.f. is

$$E_b = E_f - E_s = K(\phi_f - S\phi_t) \qquad (45)$$

$$E_b = K\phi_f(1 - S^2) \qquad \text{Eq. (2)} \qquad (46)$$

Since for constant frequency of supply current, F is proportional to ϕ_f we may write $\phi_f = CF$, C being a constant depending on the number of field turns.

$$\phi_f = CF = \frac{CE}{\sqrt{1 + S^2n^2}} \qquad \text{Eq. (16)} \qquad (47)$$

hence

$$E_b = \frac{KCE(1 - S^2)}{\sqrt{1 + S^2n^2}} \qquad (48)$$

which becomes zero at $\pm S = 1$, that is at synchronism when operated as either a motor or a generator. Above synchronism E_b increases rapidly with increase of speed.

The friction loss can best be taken into account by considering the friction torque as constant ($= d$) and subtracting this value from the delivered electrical torque so that the active mechanical torque becomes,

$$\text{Torque} = D - d \qquad (49)$$

While the effect of the iron loss is relatively small as concerns the electrical characteristics of the machine it is obviously incorrect to neglect it when determining the efficiency. For purpose of analysis it is convenient to divide the core material into three parts, the armature the field and the transformer portions. Since the frequency of the reversal of the flux in both the transformer and the field portions is constant the losses therein will depend only upon the flux. Thus considering hysteresis only, the transformer iron loss is

$$H_t = L\phi_t^{1.6} \qquad (50)$$

where L is a constant depending upon the mass of the core material similarly the field iron loss is

$$H_f = M\phi_f^{1.6} \qquad (51)$$

M being a constant

$$H_t + H_f = \phi_f^{1.6}(M + SL) \qquad \text{Eq. (2)} \qquad (52)$$

Since both the field and the transformer fluxes pass through the armature core and these two fluxes are of the same frequency but displaced in quadrature both in mechanical position and in

time-phase relation, the resultant is an elliptical field revolving always at synchronous speed, having one axis in line with the transformer and the other in line with the field, the values being $\sqrt{2}\,\phi_t$ and $\sqrt{2}\,\phi_f$ respectively. The value of the two axes may be writen thus

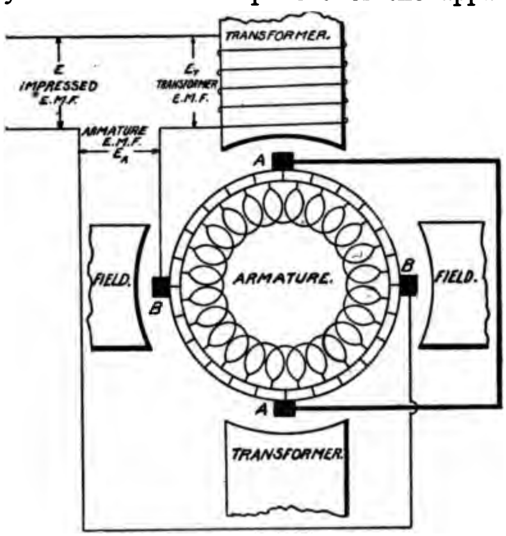
$$\sqrt{2}\,S\phi_f \text{ and } \sqrt{2}\,\phi_f$$

At synchronous speed of the armature the two become equal and since no portion of the iron is then subjected to reversal of magnetism the iron loss of the armature core is of zero value. At other speeds, while the revolving elliptical field yet travels synchronously, the armature does not travel at the same speed, so that certain sections of the armature core are subjected to fluctuations of magnetism while others are subjected to complete reversals, the sections continually being interchanged. It is due to this fact that no correct equation can be formed to represent the core loss of the armature at all speeds, since the behavior of iron when subjected to fluctuating magnetism cannot be reduced to a mathematical expression.

ALTERNATING CURRENT COMMUTATOR MOTORS.
II. REPULSION-SERIES MOTOR.*

A. S. M'ALLISTER.

A type of motor closely related to the repulsion machine in the performance of its magnetic circuits is the compensated series motor shown in Fig. 4. Its electrical circuits seem to be those of a series machine with the addition of a second set of brushes, AA, placed in mechanical line with the field coil and short-circuited upon themselves. The transformer action of this closed circuit is such that the real power which the motor receives is transmitted to the armature through this set of brushes, while the remaining set, BB, which in the plain series motor receives the full electrical power of the machine, here serves to supply only the wattless component of the apparent power.



TWO POLE MODEL OF IDEAL REPULSION-SERIES MOTOR.
FIG.-4.

This complete change in the inherent characteristics of the series machine by the mere addition of two brushes renders the study of this type of motor especially interesting.

For purpose of analysis, assume an ideal motor without resistance or local leakage reactance and consider first the conditions

* Abstract of thesis for Ph.D. degree, Cornell University.

when the armature is at rest. When a certain e.m.f., E, is impressed upon the motor terminals, the counter magnetizing effect of the current in the brush circuits, AA, is such that the e.m.f. across the transformer coil is of zero value, while that across the armature is E, Thus when $S = 0$, letting $E_t =$ transformer e.m.f. and $E_a =$ armature e.m.f.,

$$E_t = 0, \text{ and } E_a = E \qquad (53)$$

It is evident also that when $S = O$ the flux through the armature in line with the brushes AA will be of zero value, so that

$$\phi_t = 0 \qquad (54)$$

Let ϕ_f be the flux through the armature in line with the brushes BB. This flux, neglecting hysteretic effects, is in time-phase with the line current and produces by its rate of change through the armature turns a counter e.m.f. of value $E_f = E$, giving to the armature circuit a reactance when stationary, of X. The relation which exists between the flux, the frequency, and the number of armature turns can be expressed thus,

$$E_f = \frac{2 \pi f N \phi_m}{\sqrt{2}\, 10^8} \qquad (55)$$

where

$f =$ frequency in cycles per second
$N =$ effective number of armature turns
$\phi_m =$ maximum value of flux.

If C be the actual number of conductors on the armature, the actual number of turns will be $\frac{C}{2}$. These turns are evenly distributed over the surface of the armature, so that any flux which passes through the armature core will generate in each individual turn an e.m.f. proportional to the product of the cosine of the angle of displacement from the position giving maximum e.m.f. and the value of the maximum e.m.f. generated by transformer action in the position perpendicular to the flux, or the average e.m.f. per turn will be $\frac{2}{\pi}$ times the maximum. The $\frac{C}{2}$ turns are connected in continuous series, the e.m.f. in each half adding in parallel to that in the other half, so that the effective series turns equal $\frac{C}{4}$. Thus, finally

$$N = \frac{2}{\pi}\frac{C}{4} = \frac{C}{2\pi} \qquad (56)$$

$$\text{and } E_f = \frac{C\,\phi_m f}{\sqrt{2}\ 10^8} \tag{57}$$

The value of the reactance will depend inversely upon the reluctance of the paths through which the armature current must force the flux. The major portion of the reluctance is found in the air-gap, and with continuous core material and uniform air-gap around the core, the reluctance will be practically constant in all directions and will be but slightly affected by the change in specific reluctance of the core material, provided magnetic saturation is not reached. In the following discussion it will be assumed that the reluctance is constant in the direction of both sets of brushes, and that the core material on both the stator and rotor is continuous.

When dealing with shunt circuits it is convenient to analyze the various components of the current at constant e.m.f., or assuming an e.m.f. of unity, to analyze the admittance and its components. When series circuits are being considered, however, the most logical method is to deal with the e.m.f.'s for constant current, or to assume unit value of current and analyze the impedance and its various components. In accordance with the latter plan, it will be assumed initially that one ampere flows through the main motor circuits at all times and the various e.m.f.'s (impedances) will thus be investigated.

An inspection of Fig. 4 will show that one ampere through the armature circuit by way of the brushes *BB* will produce a definite value of flux independent of any changes in speed of the rotor, since there is no opposing magneto-motive force in any inductively related circuit. From this fact it follows that on the basis of unit line current ϕ_a has a constant effective value, although varying from instant to instant according to an assumed sine law. As will appear latter, while both the current through the armature and the flux produced thereby have unvarying, effective values and phase positions, the apparent reactance of the armature is not constant, but follows a parabolic curve of value with reference to change in speed.

When the armature travels at any certain speed the conductors cut the flux which is in line with the brushes *BB* and there is generated at the brushes *AA* an electro-motive force proportional at each instant to the flux ϕ_f and hence in time-phase with ϕ_f, or with the armature current through *BB*.

Let ϕ_m = maximum value of ϕ_f, then the maximum value of the e.m.f. generated at AA due to dynamo speed action will be,

$$E_m = \frac{C \phi_m V}{10^8} \tag{58}$$

where V is revolutions per second. The virtual value of this electro-motive force will be

$$E_v = \frac{C \phi_m V}{\sqrt{2}\ 10^8} \tag{59}$$

A comparison of (59) and (57) will show that at a speed V revolutions per second such that $V = f$ in cycles per second, $E_v = E_f$ for any value of ϕ_m. Consequently, the speed e.m.f. due to any flux threading the armature turns, at synchronism becomes equal to the transformer e.m.f. due to the same flux through the same turns. E_f is in time-quadrature and E_v in time-phase with the flux at any speed, hence, E_v is in time-quadrature with E_f or in time-phase with the line current.

The brushes AA remain at all times connected directly together by conductor of negligible resistance so that the resultant e.m.f. between the brushes must remain of zero value. On this account when an e.m.f. E_v is generated between the brushes by dynamo speed action, a current flows through the local circuit giving a magneto-motive force such that the flux produced thereby generates in the armature conductors by its rate of change, an e.m.f. equal and opposite to E_v. This flux, ϕ_t, is proportional to E_v and being in time-quadrature thereto, is in time-phase with E_f, or in time-quadrature with ϕ_f.

From the transformer relations it is seen that

$$E_v = \frac{C\sqrt{2}\ \phi_t f}{\sqrt{2}\ 10^8} \tag{60}$$

where $\sqrt{2}\ \phi_t$ is maximum value of flux due to current through brushes AA. See (57).

$$E_v = G\,\phi_t \tag{61}$$

where G is a proportionality constant.

Let S be the speed, with synchronism as unity, then

$$E_v = S\,E_f \tag{62}$$

and

$$\phi_t = S\,\phi_f, \tag{63}$$

effective values being used. This is the fundamental magnetic equation of the repulsion-series motor.

Flux ϕ_t passes through the transformer turns on the stator in line with the brushes AA as shown in Fig. 4 and generates therein by its rate of change an e.m.f. E_r such that

$$E_t = n\,E_v \qquad (64)$$

where n is the ratio of effective transformer to armature turns. This e.m.f. is in phase with E_v, in quadrature with E_r and hence is in phase opposition with the line current and produces the effect of apparent resistance in the main motor circuits.

Combining (62) and (64)

$$E_t = S\,n\,E_r \qquad (65)$$

Since E_r is the transformer e.m.f. in the armature circuits due to constant effective value of flux from one ampere, we may write

$$E_r = X \qquad (66)$$

where X is the stationary reactance of the armature circuit, so that the apparent resistance of the transformer circuit is

$$\underline{R} = S\,n\,X \qquad (67)$$

Under speed conditions the armature conductors cut the flux in line with the brushes AA, and there is generated thereby an e.m.f. which appears as a maximum at the brushes BB. This e.m.f. is in phase with ϕ_t, in quadrature with ϕ_r and in phase opposition to E_r. If E_s be the value of this e.m.f. we may write,

$$E_s = \frac{C\sqrt{2}\,\phi_t\,S}{\sqrt{2}\;10^8} \qquad (68)$$

from dynamo speed relations. Comparing (60) and (68) and remembering that f is unity in terms of speed, there is obtained

$$E_s = S\,E_v \qquad (69)$$

from (62) and (66)

$$E_s = S^2\,E_r = S^2\,X \qquad (70)$$

Therefore the e.m.f. across the armature at $B\,B$ will be

$$E_a = E_r - E_s = X(1 - S^2) \qquad (71)$$

This e.m.f. is in quadrature with the line current and is in effect an apparent reactance, so that the apparent reactance of the motor circuits which is confined to the armature winding is

$$\underline{X} = X(1 - S^2) \qquad (72)$$

The apparent impedance of the motor circuits at speed S is

$$\underline{Z} = \sqrt{\underline{R}^2 + \underline{X}^2} = \sqrt{(S\,n\,X)^2 + X^2(1 - S^2)^2} \qquad (73)$$

This is the fundamental impedance equation of the ideal repulsion-series motor.

The power factor is

$$\cos\theta = \frac{R}{Z} = \frac{S\,n\,X}{\sqrt{(S\,n\,X)^2 + X^2(1 - S^2)^2}} \tag{74}$$

The line current is

$$I = \frac{E}{Z} = \frac{E}{\sqrt{S^2 n^2 X^2 + X^2(1 - S^2)^2}} \tag{75}$$

The power is,

$$P = E\,I\cos\theta = \frac{E^2\,S\,X\,n}{X^2\,S^2\,n^2 + X^2(1 - S^2)^2} \tag{76}$$

It will be noted that both the power and the power factor reverse when S is negative. Thus the machine becomes a generator when driven against its torque.

The wattless factor is,

$$\operatorname{Sin}\theta = \frac{X}{Z} = \frac{X(1 - S^2)}{\sqrt{S^2\,n^2\,X^2 + X^2(1 - S^2)^2}} \tag{77}$$

and becomes negative when S is greater than 1, so that above synchronism when operated as either a generator or motor the machine draws leading wattless current from the supply system. At $S = 1$, Sin $\theta = 0$, which means that the power factor is unity at synchronous speed, as may be seen also from eq. (74).

At $S = 0$,

$$I = \frac{E}{X}$$

At $S = 1$, $\quad I = \dfrac{E}{n\,X} \quad$ That is, at synchronism the line current is equal to the current at start divided by the ratio of transformer to armature turns. If $N = 1$, the current at synchronism is of the same value as at start but the power factor which at start was 0 has a value of 1 at synchronism. This interesting feature will be touched upon later.

The torque is

$$D = \frac{P}{S} = \frac{E^2\,n\,X}{\sqrt{X^2\,S^2\,n^2 + X^2(1 - S^2)^2}} = I^2\,n\,X \tag{78}$$

and is maximum at maximum current and retains its sign when S is reversed.

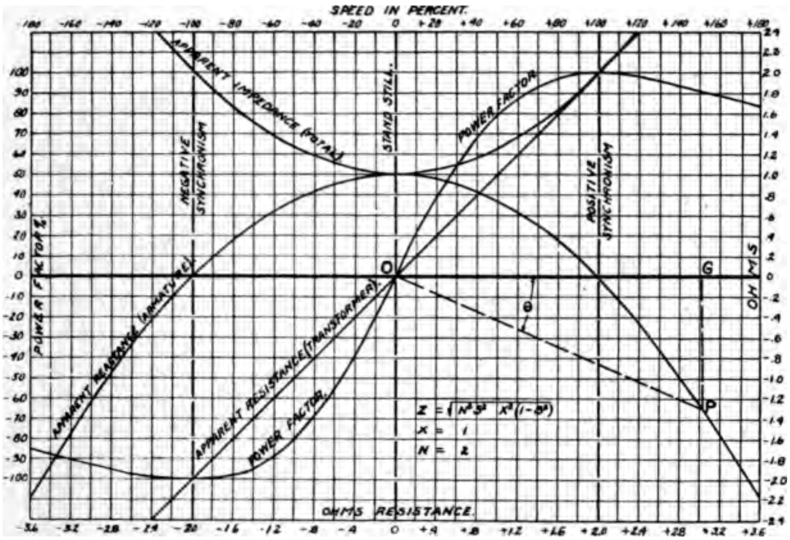
When $S = 0$ the secondary current, I_s, is nI, and is in phase

opposition with the transformer current I. See Fig. 4 When $S = 1$

$$I_s = \sqrt{n^2 I^2 + I^2} \qquad \text{and at any speed } S,$$

$$I_s = \sqrt{n^2 I^2 + S^2 I^2} = I\sqrt{n^2 + S^2} \tag{79}$$

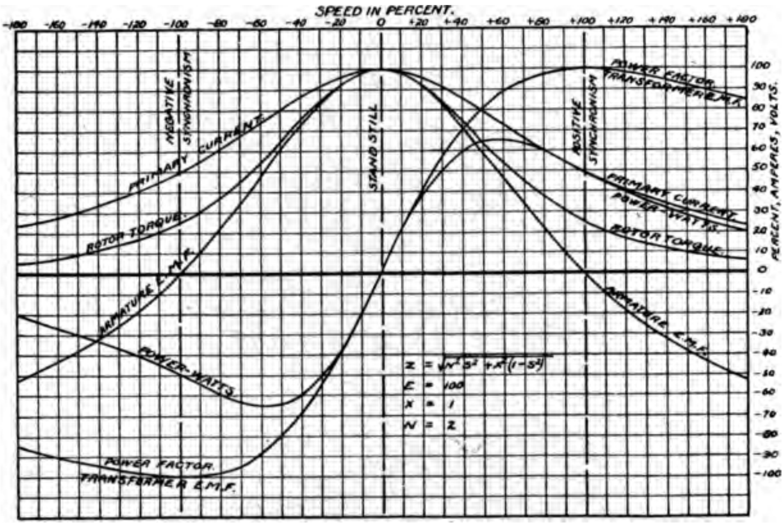
$$I_s = \frac{E\sqrt{n^2 + S^2}}{\sqrt{S^2 n^2 X^2 + X^2 (1 - S^2)^2}} \tag{80}$$



CHARACTERISTICS OF IDEAL REPULSION-SERIES MOTOR.
FIG.-5.

In Fig. 5 are shown the results of calculations for a certain ideal repulsion-series motor of which $X = 1$ and $n = 2$. It is seen that with speed as abscissa, the curve representing the apparent resistance of the motor circuits is a right line while that for the apparent reactance is a parabola. At any chosen speed the quadrature sum of these two components gives the apparent impedance of the motor. Since the scale for representing the speed is in all respects independent of that used for the apparent resistance, it is possible always so to select values for the one scale that a given distance from the origin may simultaneously represent both the resistance and the speed. This method of plotting the values leads to a very simple vector diagram for representing both the value and phase position of the apparent impedance at any speed, and for determining the power-factor from inspection. Thus at any speed such as is shown at G the dis-

tance OG is the apparent resistance, the distance GP is the apparent reactance, OP is the apparent impedance while the angle POG is the angle of lead of the primary current and its cosine is the power-factor.



CHARACTERISTICS OF IDEAL REPULSION-SERIES MOTOR.
FIG.-6.

Fig. 6 gives the complete performance charactersitics of the above ideal repulsion-series motor at various positive and negative speeds when operated at an impressed e.m.f. of 100 volts.

It will be noted that the armature e.m.f., which has a certain value at standstill, decreases with increase of speed, becomes zero at synchronism and then increases at higher speeds. The transformer e.m.f. is zero at starting, increases to a maximum at synchronism and then continually decreases with increase of speed.

The inductive portion of the impedance is contained wholly by the armature circuit, while the non-inductive is confined to the transformer coil; thus the power-factor is zero at standstill, reaches unity at syhchronism and then decreases due to the lagging component of the motor impedance (leading wattless current). In comparison with the ordinary compensated series motor whose armature e.m.f. is, for the most part, non-inductive and continually increases with increase of speed, and whose inductive field e.m.f, decreases continually with increase of speed and whose power-factor never reaches unity, the repulsion-series motor furnishes a most striking contrast. The machine resem-

bles the repulsion motor in regard to its magnetic behavior, but the performance of its electric circuits differs from that of the repulsion motor due to the fact that the speed e.m.f. introduced into the armature circuit *BB* (Fig. 4) which has been substituted for the field coil of the repulsion motor (See Fig. 2) is in a direction continually to decrease the apparent reactance of the field circuit and thus to decrease the inductive component of the impedance of the circuits and to improve the power factor and the operating characteristics. It is an interesting fact that under all conditions of operation the e.m.f. in the coils short circuited by the brushes *BB* is of zero value, so that no objectional features are introduced by substituting the armature circuit for the field coil of the repulsion motor, while the performance is materially improved. Experiments show that even with currents of many times normal value and at the highest commercial frequency no indication of sparking is found at the brushes *BB*. This feature will be treated in detail later.

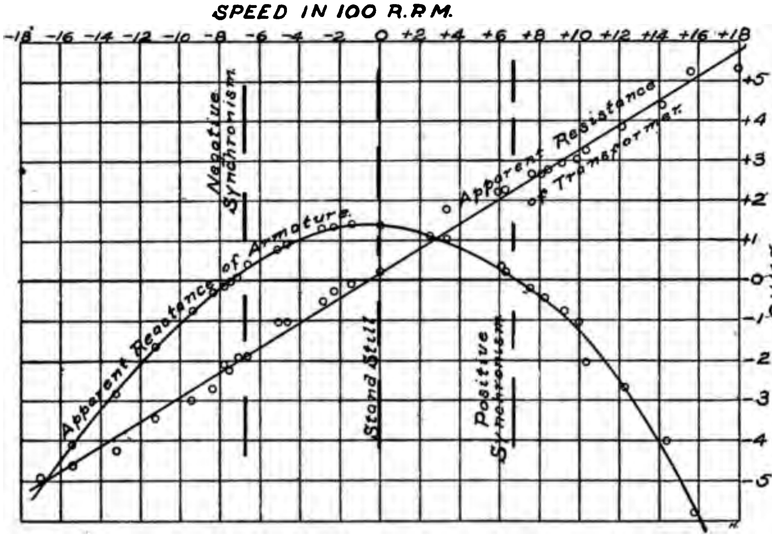
An inspection of Fig. 5 and of equation (73) will reveal the fact that at synchronism the apparent impedance is n times its value at stand still. If n be made unity, the apparent impedance at synchronism will be equal to that at stand still, while between these speeds it varies inappreciably. This means that from zero speed to synchronism the primary current varies but slightly, and that the torque, which is proportional to the square of the primary current is practically constant throughout this range of speed. These facts show that a unity ratio repulsion series-motor is a constant torque machine at speeds from negative to positive synchronism, the relative phase position of the current and the e.m.f. changing so as always to cause them to give by their vector product the power represented by the torque at the various speeds. Above synchronism the torque decreases continually, tending to disappear at infinite speed.

Any desired torque-speed characteristic within limits can be obtained by giving to n a corresponding value, the torque at synchronism being equal to the starting torque divided by the square of the ratio of transformer to armature turns.

In connection with the discussion of the expression for determining the value of the torque it is well to mention the fact that the commonly accepted explanations as to the physical phenomena involved in the production of torque must be somewhat modified if actual conditions of operation known to exist are to

be represented. Referring to Fig. 4, it will be noted that when the armature is stationary there exists no magnetism in line with the brushes *AA*, so that the current which enters the armature by way of the brushes *BB* could not be said to produce torque by its product with magnetism in mechanical quadrature with it. Similarly, the flux in line with the brushes *BB* could not be said to be attracted or repelled by magnetism which does not exist. That the current through *AA* produces torque by its product with the magnetism due to current through *BB* would be contrary to accepted methods of reasoning, since both currents flow in the same structure, yet, as concerns the torque, the effect is quite the same as though the flux in line with the brushes *BB* were due to current in a coil located on the field core. (As shown in Fig. 2 for the ordinary repulsion motor.)

The calculated impedance characteristics shown in Fig. 5 are based on arbitrarily assumed constants of a repulsion-series motor under ideal conditions. It is obviously impossible to obtain such characteristics from an actual motor, since all losses

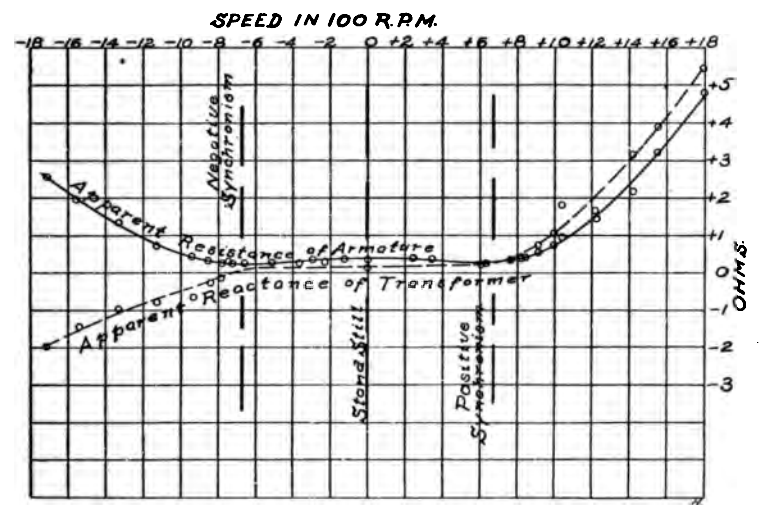


TEST OF REPULSION SERIES MOTOR—ACTIVE FACTORS OF OPERATION
FIG. 7.

and minor disturbing influences have been neglected in determining the various values. As a check upon the theory given above, the curves of Figs. 7 and 8, as obtained from tests of a repulsion-series motor, are presented herewith. It will be ob-

served that the apparent resistance of the transformer coil varies directly with the speed and becomes negative at negative speed, while the apparent reactance of the armature decreases with increase of speed in either direction and, following approximately a parabolic law, reverses and becomes negative at speeds slightly in excess of synchronism. A comparison of the general shape of the curves of Fig. 7 and Fig. 5 will show to what extent the assumed ideal conditions can be realized in practice, and it would indicate that, as concerns the active factors of operation, the equations given represent the facts involved. The neglect of the local resistance of the transformer circuit leads to the discrepancy between the theoretical and observed curves as found at zero speed, the latter curve indicating a certain apparent resistance when the armature is stationary. Similarly at synchronous speed the observed apparent reactance of the armature is not of zero value due to the local leakage reactance of the circuit.

In the determination of the theoretical curves only active factors have been considered, and it has been shown that the apparent reactance of the motor circuits is confined to the armature, while the e.m.f. counter generated in the transformer coil gives the effect of apparent resistance located exclusively within this coil. The neglected disturbing factors, the apparent resistance of the armature and the apparent reactance of the transformer, are of relatively small and practically constant value



TEST OF REPULSION SERIES MOTOR–DISTURBING FACTORS OF OPERATION.
FIG. 8.

throughout the operating range of speed from negative to positive synchronism, but they become of prime importance when the speed exceeds this value in either direction, as shown by the curves of Fig. 8 obtained from the test of a repulsion-series motor giving the curves of Fig. 7. The predominating influence of the disturbing factors above synchronism is attributable largely to the effect of the short circuit by the brushes *AA* (Fig. 4) of coils in which there is produced an active e.m.f. by combined transformer and speed action. This short circuiting effect will be treated in detail later.

The resistance and local leakage reactance of the coils may be included in the theoretical equations as follows:

Let R_t = resistance of transformer coil
R_a = resistance of armature circuit
R_s = resistance of secondary circuit
X_t = leakage reactance of transformer
X_a = leakage reactance of armature
X_s = leakage reactance of secondary circuit

then copper loss of motor circuits will be

$$I^2 (R_t + R_a) + I_s^2 R_s \qquad (81)$$

$$I_s = I \sqrt{n^2 + S^2} \qquad (82)$$

$$I^2 [R_t + R_a + (n^2 + S^2) R_s] = I^2 R_m \qquad (83)$$

where R_m is the effective equivalent value of the motor circuit resistance, that is,

$$R_m = R_t + R_a + (n^2 + S^2) R_s \qquad (84)$$

Similarly it may be shown that the effective equivalent value of the leakage reactance of the motor circuits is

$$X_m = X_t + X_a + (n^2 + S^2) X_s \qquad (85)$$

combining equations (84) and (85) with (73) the expression for the apparent impedance of the motor circuits becomes

$$\underline{Z} = \sqrt{(\underline{R} + R_m)^2 + (\underline{X} + X_m)^2} = \qquad (86)$$

$$\sqrt{[S n X + R_t + R_a + (n^2 + S^2) R_s]^2 + [X(1 - S^2) + X_t + X_a + (n^2 + S^2) X_s]^2}$$

$$I = \frac{E}{\sqrt{(S n X + R_m)^2 + [X(1 - S^2) + X_m]^2}} \qquad (87)$$

$$\cos \theta = \frac{S n X + R_m}{\sqrt{[X(1 - S^2) + X_m]^2 + (S n X + R_m)^2}} \qquad (88)$$

$$\text{Input} = E\,I\cos\theta \tag{89}$$

$$\text{Output} = E\,I\cos\theta - I^2 R_m = P \tag{90}$$

$$P = \frac{E^2 (S\,n\,X + R_m)}{(S\,n\,X + R_m)^2 + [X(1 - S^2) + X_m]^2} - \frac{E^2 R_m}{(S\,n\,X + R_m)^2 + [X(1 - S) + X_m]^2} \tag{91}$$

$$P = \frac{E^2 S\,n\,X}{(S\,n\,X + R_m)^2 + [X(1 - S^2 + X_m]^2} = I^2 S\,n\,X \tag{92}$$

$$\text{torque} = D = \frac{P}{S} = I^2 n\,X \tag{93}$$

The above equations, though incomplete on account of neglecting the brush shortening effect and the magnetic losses in the cores, represent quite closely the electrical characteristics of the repulsion-series motor when operated between negative and positive synchronism, throughout which range of speed the disturbing factors are of secondary importance.

The e.m.f. in the coils short circuited by the brushes can be treated by a method similar to that used with the repulsion motor. Referring to Fig. 4, the coil under the brush A is subjected to the transformer effect of the flux, ϕ_r, in line with the brushes, BB, and the dynamo speed effect of the flux, ϕ_t, in line with the brushes AA.

Effective values being used throughout, the transformer e.m.f. will be, assuming C actual conductors on the armature,

$$e_t = \frac{\frac{C\pi}{2} f\sqrt{2}\,\phi_r}{\sqrt{2}\,\frac{C}{4}\,10^8} = \frac{\pi f \phi_r}{2\cdot 10^8} \tag{94}$$

in volts for one coil. See equation (57). This e.m.f. is in time-quadrature with ϕ_r.

The dynamo speed e.m.f. in volts for one coil will be,

$$e_s = \frac{\frac{C\pi}{2} V\sqrt{2}\,\phi_t}{\sqrt{2}\,\frac{C}{4}\,10^8} = \frac{\pi\,V\phi_t}{2\cdot 10^8} \tag{95}$$

See equation (59). This e.m.f. is in time-phase with ϕ_t and hence is time-quadrature with ϕ_r. Thus the electro-motive force in the coil under the brush A is

$$E_a = e_t - e_s = \frac{\pi}{2\cdot 10^8}(f\phi_r - V\phi_t) \tag{96}$$

But $V = f S$ and $\phi_t = S \phi_f$ (97)

See equation (63), hence

$$V \phi_t = S^2 f \phi_f, \tag{98}$$

so that

$$E_a = \frac{\pi f \phi_f}{2 \cdot 10^8} (1 - S^2) \tag{99}$$

This resultant electromotive force has a value at standstill, when S is zero, of

$$\frac{\frac{\pi}{2} E_f}{\frac{C}{4}} = \frac{I \pi X}{2 C} \tag{100}$$

See equation (66) Thus

$$\text{finally, } E_a = \frac{I \pi X}{2 C} (1 - S^2) \tag{101}$$

When the armature is stationary the electro-motive force in the coil short circuited by the brush A has the value given by equation (100), which, with any practical motor, is of sufficient value to cause considerable heating if the armature remains at rest, or to produce a fair amount of sparking as the armature starts in motion. At synchronous speed, however, this electro-motive force disappears entirely, and the performance of the machine as to commutation is perfect. As the speed exceeds this critical value in either the positive or negative direction, the electro-motive force in the short-circuited coil increases rapidly, resulting in a return in an augmented form of the sparking found at lower speeds and producing the disturbing factors shown by the curves of Fig. 8.

Since the e.m.f. in the coil under the brush A reduces to zero at both positive and negative synchronism and reverses with reference to the time-phase position of the line current at speeds exceeding synchronism in either direction, it possesses at high speeds the same time-phase position when the machine is operated as a generator as when it is used as a motor. The time-phase of its reactive effect upon the current which flows in the armature through the brushes BB is of the same sign at high positive and negative speeds, but reversed from the phase position of the effect at speeds below synchronism. A study of the test curves of Fig. 8 will show the magnitude of these effects, and the

reversal of their time-phase positions in accordance with the theoretical considerations.

With reversal of direction of rotation the time-phase position of the flux threading the transformer coil (Fig. 4) reverses with reference to the line current, and hence in its reactive effect upon the transformer flux the current in the coil short circuited by the brush A becomes negative at speeds above negative synchronism, though positive above synchronism in the positive direction. At speeds below synchronism, when the flux is large the e.m.f. is small, and *vice versa*, so that the reactive effect is in any case relatively small and of more or less constant value. See Fig. 8.

It will be noted that in analyzing the disturbing factors no account has been taken of the short-circuiting effect at the brushes BB, Fig. 4. This treatment is in accord with the statement previously made that the component e.m.f.s generated in the coils under these brushes are at all times of values such as to render the resultant zero. The proof of this fact is as follows:

The transformer e.m.f. in the coil under B due to flux, ϕ_t, in line with brushes AA is

$$e_f = \frac{\pi f \phi_t}{2 \cdot 10^8} \qquad (102)$$

See equation (94). This e.m.f. is in time-quadrature with ϕ_t.

The dynamo speed e.m.f. is

$$e_v = \frac{\pi V \phi_f}{2 \cdot 10^8} \qquad (103)$$

This e.m.f. is in time-phase with ϕ_f, in time quadrature with ϕ_t, and is in phase opposition to e_f. Thus the resultant e.m.f. is

$$E_b = e_f - e_v = -\frac{\pi}{2 \cdot 10^8}(f \phi_t - V \phi_f) \qquad (104)$$

Since $V = f S$ and $\phi_t = S \phi_f$ from equations (97) and (63),

$$f \phi_t = V \phi_f \qquad (105)$$

and

$$E_b = 0 \qquad (106)$$

This theoretical deduction is substantially corroborated by experimental evidence, as has been noted above. Even upon superficial examination such a result is to be expected, since the vector sum of all e.m.f.s in the armature in mechanical line with the short-circuited brushes AA must be zero, while the e.m.f. in the coil at brush B must equal its proper share of this e.m.f. or

$$E_b = \frac{0 \cdot \pi}{C\,2} = 0 \qquad (107)$$

A similar course of reasoning allows of the determination of the electro-motive force under the brush A. See equation (71).

$$E_a = \frac{E_a \pi}{C\,2} = \frac{\pi X (1 - S^2)}{2 \cdot C} \qquad (108)$$

for unit current. For I amperes this becomes.

$$E_a = \frac{I \pi X}{2 \cdot C \cdot} (1 - S^2) \qquad (109)$$

See equation (101).

From the facts just indicated it would seem that perfect commutation dictates that the electro-motive force across a diameter ninety electrical degrees from the brushes upon the armature be at all times of zero value. Methods for approximating this condition will be discussed in a later paper.

It has been stated that the magnetic circuits of the repulsion-series motor are quite the same as those of the repulsion motor. The fluxes in line with the two brush circuits under all conditions are in time-quadrature and have relative values varying with the speed such that at all times

$$\phi_t = S\,\phi_f \qquad (110)$$

There exists, therefore, at all speeds a revolving magnetic field elliptical in form as to space representation. At standstill the ellipse becomes a straight line in the direction of the brushes BB (Fig. 4), at infinite speed in either direction the ellipse would again be a straight line in the direction of the brushes AA, while at either positive or negative synchronism the ellipse is a true circle, the instantaneous maximum value of the revolving magnetism traveling in the direction of motion of the armature. At synchronous speed, therefore, the magnetic losses in the armature core disappear, while the losses in the stator core are evenly distributed around its circumference.

ALTERNATING CURRENT COMMUTATOR MOTORS.
III. COMPENSATED SERIES MOTORS.*

A. S. M'ALLISTER.

The combined transformer and motor features of commutator type of alternating current machinery are well exemplified in the plain series motor as illustrated in Fig. 9. When the rotor is

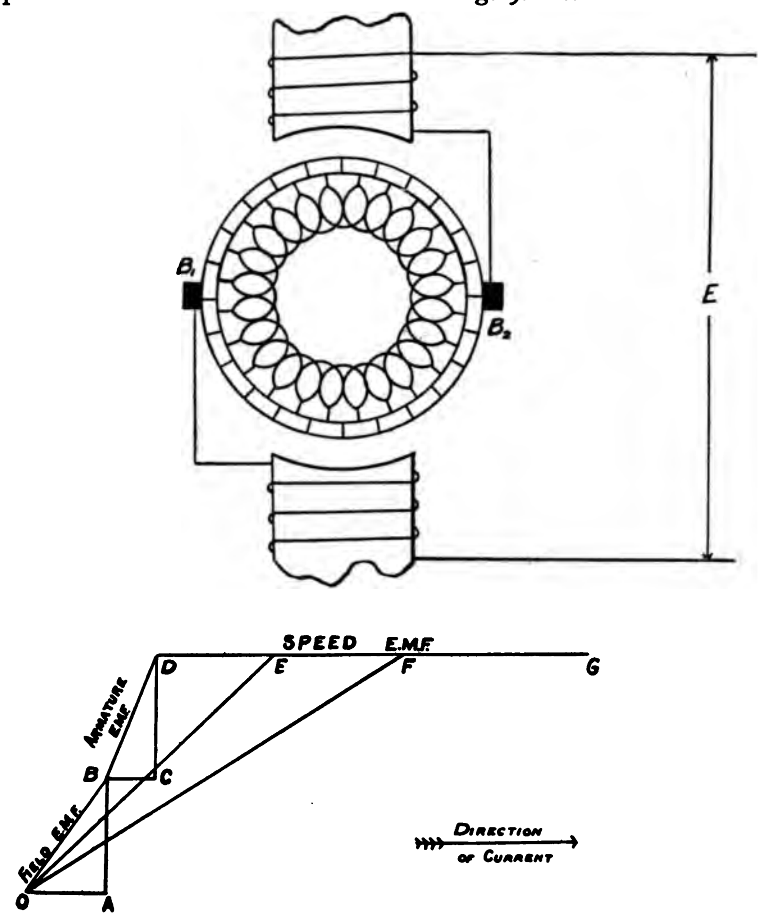


FIG. 9. Plain Series Motor.

stationary, the field and armature circuits of the motor form two impedances in series. Assuming initially an ideal motor without

* Abstract of thesis for Ph.D. degree, Cornell University.

resistance and local leakage reactance, each impedance consists of pure reactance, the current in the circuit having a value such that its magneto-motive force when flowing through the armature and field turns causes to flow through the reluctance of the magnetic path that value of flux the rate of change of which generates in the windings an electro-motive force equal to the impressed.

If E be the impressed e.m.f., E_f the counter transformer e.m.f. across the field coil and E_a the counter transformer e.m.f. across the armature coil, when the armature is stationary

$$E = E_f + E_a \qquad (111)$$

From fundamental transformer relations there is obtained the equation

$$E_f = \frac{2\pi f N_f \phi_f}{\sqrt{2}\, 10^8}, \text{ see eq. (53)} \qquad (112)$$

where f = frequency in cycles per second

N_f = effective number of field turns

ϕ_f = maximum value of field flux.

Similarly

$$E_a = \frac{2\pi f N_a \phi_a}{\sqrt{2}\, 10^8} \qquad (113)$$

where N_a = effective number of armature turns

ϕ_a = maximum value of armature flux.

Since the field and armature circuits are electrically series connected and are mechanically so placed as not to be inductively related, with uniform reluctance around the air gap the fluxes in mechanical line with the two circuits being due to the magneto-motive force of the same current will be proportional to the effective number of turns on the two circuits.

Therefore

$$\frac{\phi_f}{N_f} = \frac{\phi_a}{N_a} \qquad (114)$$

If n be the ratio of effective field to armature turns

$$N_f = n\, N_a \qquad (115)$$

and

$$\phi_f = n\, \phi_a \qquad (116)$$

Let C be the actual number of conductors on the armature, then

$$N_a = \frac{C}{2\pi} \text{ (see eq. 56)} \qquad (117)$$

Under speed conditions the armature conductors cut the field magnetism and there is generated by dynamo action a counter

e.m.f. proportional to the product of the field flux and the speed, in time-phase with the flux, in leading time quadrature with the field e.m.f., E_f and the armature e.m.f. E_a and in phase opposition with the current.

Thus

$$E_v = \frac{C \phi_f V}{\sqrt{2}\, 10^8} \text{ (see eq. 59)} \qquad (118)$$

where V is revolutions per second of bipolar model.

Combining (117) and (118)

$$E_v = \frac{2\pi V N_a \phi_f}{\sqrt{2}\, 10^8} \qquad (119)$$

If S be the speed with synchronism as unity, then

$$V = Sf \qquad (120)$$

and

$$E_v = \frac{2\pi S f N_a \phi_f}{\sqrt{2}\, 10^8} \qquad (121)$$

combining (113) (116) and (121)

$$E_v = \frac{E_a S \phi_f}{\phi_a} = E_a S n \qquad (122)$$

combining (112) (115) and (121)

$$E_v = \frac{E_f S N_a}{N_f} = \frac{E_f S}{n} \qquad (123)$$

comparing (122) and (123)

$$E_f = n^2 E_a$$

Under speed conditions the impressed e.m.f. is balanced by three components, E_v in time phase opposition with the line current and E_f and E_a, both in leading time quadrature with the line current.

Thus

$$E = \sqrt{E_v^2 + (E_a + E_f)^2} \qquad (125)$$

$$E = \sqrt{E_a^2 S^2 n^2 + (E_a + n^2 E_a)^2} = E_a \sqrt{S^2 n^2 + (1 + n^2)^2} \qquad (126)$$

This is the fundamental electro-motive force equation of the plain series motor having uniform reluctance around the air-gap.

On the basis of unit line current the electro-motive forces may be treated as impedances, as was done with the repulsion-series motor, so that the impedance equation becomes

$$\underline{Z} = X_a \sqrt{S^2 n^2 + (1 + n^2)^2} \qquad (127)$$

where $S n X_a = \underline{R}$ and $(1 + n^2) X_a = \underline{X}$

The power factor is

$$\frac{R}{Z} = \cos\theta = \frac{Sn}{\sqrt{S^2n^2 + (1 + n^2)^2}} \qquad (128)$$

which reverses when S becomes negative and continually approaches unity with increase of S in either direction.

When $S = 1$, or at synchronism

$$\cos\theta = \frac{n}{\sqrt{n^2 + (1 + n^2)^2}} \qquad (129)$$

which when $n = 1$ or for unity ratio of field to armature turns becomes

$$\cos\theta = \frac{1}{\sqrt{1 + (1 + 1^2)}} = .447, \qquad (130)$$

and decreases with either an increase or decrease of n. It is apparent therefore that the power factor of such a machine is inherently very low and cannot be improved by a mere change in the ratio of field to armature turns.

The line current is

$$I = \frac{E}{Z} = \frac{E}{X_a} \cdot \frac{1}{\sqrt{S^2n^2 + (1 + n^2)^2}} \qquad (131)$$

The power is

$$P = EI\cos\theta = \frac{E^2}{X_a} \cdot \frac{Sn}{S^2n^2 + (1 + n^2)^2} \qquad (132)$$

which becomes negative when S reverses, or the machine operates as a generator when driven against its natural tendency to rotation.

The torque is

$$D = \frac{P}{S} = \frac{E^2}{X_a} \cdot \frac{n}{S^2n^2 + (1 + n^2)^2} = I^2 n X_a. \qquad (133)$$

which is maximum at maximum current and retains its sign when S is reversed.

At starting the torque is

$$D_0 = \frac{E^2}{X_a} \cdot \frac{n}{(1 + n^2)^2} \qquad (134)$$

At synchronous speed, the torque is

$$D_s = \frac{E^2}{X_a} \quad \frac{n}{n^2 + (1 + n^2)^2} \qquad (135)$$

and

$$\frac{D_s}{D_0} = \frac{(1 + n^2)^2}{n^2 + (1 + n^2)^2} \qquad (136)$$

which when n is negligibly small approaches a value of unity and when n is infinitely large also tends to reach a value of unity. When $n = 1$ equation (136) reduces to

$$\frac{D_s}{D_0} = \frac{(2)^2}{1 + (2)^2} = .8 \tag{137}$$

ths interpretation of which is that the torque of the unity-ratio single-phase, plain series motor with uniform reluctance around the air-gap varies only 20 per cent. from standstill to synchronism, and therefore, that such a machine is unsuited for traction. This statement applies to the ideal single-phase motor without internal losses and must be somewhat modified to include true operating conditions. The method of treating the various losses has previously been discussed and will further be enlarged upon in connection with the compensated types of series machines. A little consideration will show that such modifications as must be introduced have a detrimental effect upon the characteristics of the machine, and tend to lay greater stress upon the statement just made. These facts are graphically represented in the performance (impedance) diagram of Fig. 9. OA is the power and AB the reactive component of the apparent field impedance at starting while BC and CD are the corresponding power and reactive components of the apparent armature impedance. The power component of apparent armature impedance due to dynamo speed action is shown as DE or DF giving the resultant impedance under speed conditions of OE or OF and indicating an angle of lag of the circuit current behind the impressed e.m.f. of EOA or FOA. The variation in torque due to increase of speed from synchronism to double synchronism with a unity ratio constant reluctance machine, as represented in Fig. 9, would be as the square of the ratio of OF to OE.

An inspection of equation (136) will reveal the fact that a a change in the value of n does not improve the torque characteristics of the machine unless such change be accompanied with an increase in reluctance of the magnetic structure in line with the brushes B_1 B_2 (Fig. 9). That is to say, if the mechanical construction is such that equation (114) may be written

$$\frac{\phi_f}{N_f} = m\,\frac{\phi_a}{N_a} \tag{138}$$

where m is a constant of a value many times unity, the operating characteristics of the machine become much improved.

Thus equation (116) becomes

$$\phi_f = m\,n\,\phi_a \tag{139}$$

and equation (122) is changed to

$$E_v = \frac{E_a S \phi_f}{\phi_a} = E_a S m n = \frac{E_f S}{n} \tag{140}$$

$$E_a = \frac{E_f}{m n^2} \tag{141}$$

$$E = \sqrt{E_v^2 + (E_a + E_f)^2} = \sqrt{\left(\frac{E_f S}{n}\right)^2 + \left(\frac{E_f}{m n^2} + E_f\right)^2} \tag{142}$$

$$E = E_f \sqrt{\left(\frac{S}{n}\right)^2 + \left(1 + \frac{1}{m n^2}\right)^2} \tag{143}$$

$$\underline{R} = \frac{S}{n} X_f, \quad \underline{X} = X_f\left(1 + \frac{1}{m n^2}\right) \tag{144}$$

$$\text{Cos}\,\theta = \frac{R}{Z} = \frac{\frac{S}{n}}{\sqrt{\left(\frac{S}{n}\right)^2 + \left(1 + \frac{1}{m n^2}\right)^2}} = \frac{S}{\sqrt{S + \left(\frac{mn^2+1}{mn}\right)}} \tag{145}$$

when $S = 1$ or at synchronism

$$\text{Cos}\,\theta = \frac{\frac{1}{n}}{\sqrt{\left(\frac{1}{n}\right)^2 + \left(\frac{m n^2 + 1}{m n}\right)^2}} = \frac{1}{\sqrt{1 + \left(\frac{m n^2 + 1}{m n^2}\right)^2}} \tag{146}$$

With an excessively large reluctance of the magnetic structure in line with the brushes $B_1 B_2$ (Fig. 9), that is, with an enormous value of m, the power factor at synchronous speed approaches

$$\text{Cos}\,\theta = \frac{1}{\sqrt{1 + n^2}} \tag{147}$$

the interpretation of which equation is that the operating power-factor of such a machine is largely dependent upon the ratio of field to armature turns. A little study will show that at any chosen speed, whether synchronous or not, the cotangent of the angle of lag is directly proportional to the ratio of armature to field turns, and that the power-factor, the corresponding cosine, can be given any desired value by a proper proportioning of the windings. This feature will be treated more in detail when dealing with compensated motors.

The current of the high brush-line-reluctance machine is

$$I = \frac{E}{Z} = \frac{E\,n}{X_r} \cdot \frac{1}{\sqrt{S^2 + \left(\frac{m\,n^2 + 1}{m\,n}\right)^2}} \tag{148}$$

The power is

$$P = E\,I \cos\theta = \frac{E^2 n\,S}{X_r} \cdot \frac{1}{S^2 + \left(\frac{m\,n^2 + 1}{m\,n}\right)^2} \tag{149}$$

The torque is

$$D = \frac{P}{S} = \frac{E^2 n}{X_r} \cdot \frac{1}{S^2 + \left(\frac{m\,n^2 + 1}{m\,n}\right)^2} = \frac{I^2 X_r}{n} \tag{150}$$

At starting the torque is

$$D_0 = \frac{E^2 n}{X_r \left(\frac{mn^2 + 1}{m\,n}\right)^2} \tag{150}$$

At synchronous speed the torque is

$$D_s = \frac{E^2 n}{X_r \left[1 + \left(\frac{m\,n^2 + 1}{m\,n}\right)^2 \right]} \tag{151}$$

$$\frac{D_s}{D_0} = \frac{\left(\frac{m\,n^2 + 1}{m\,n}\right)^2}{1 + \left(\frac{m\,n^2 + 1}{m\,n}\right)^2} \tag{152}$$

which ratio, with an enormous value of m, approaches

$$\frac{D_s}{D_0} = \frac{n^2}{1 + n^2} \tag{153}$$

the significance of which is that the change of torque from standstill to synchronism can be altered at will by change in the ratio of field to armature turn and that a relatively low value of n would produce a machine suitable for traction.

By using projecting field poles thus leaving large air-gaps in the axial brush line and thereby increasing the reluctance of the structure in line with the magneto-motive force of the armature current, the flux produced by the armature current may be materially reduced, thus giving to m a relatively large value, and the power factor will be thereby correspondingly increased with a resultant improvement in the torque characteristics of the machine. Even under the most favorable conditions, however, it is impossible to reduce the reactance of the armature circuit to an inappreciable

value, that is, to give to m an enormous value, dne to the inevitable presence of the magnetic material of the projecting poles.

The most satisfactory method of reducing the inductive effect of the armature current is to surround the revolving armature winding with properly disposed stationary conductors through

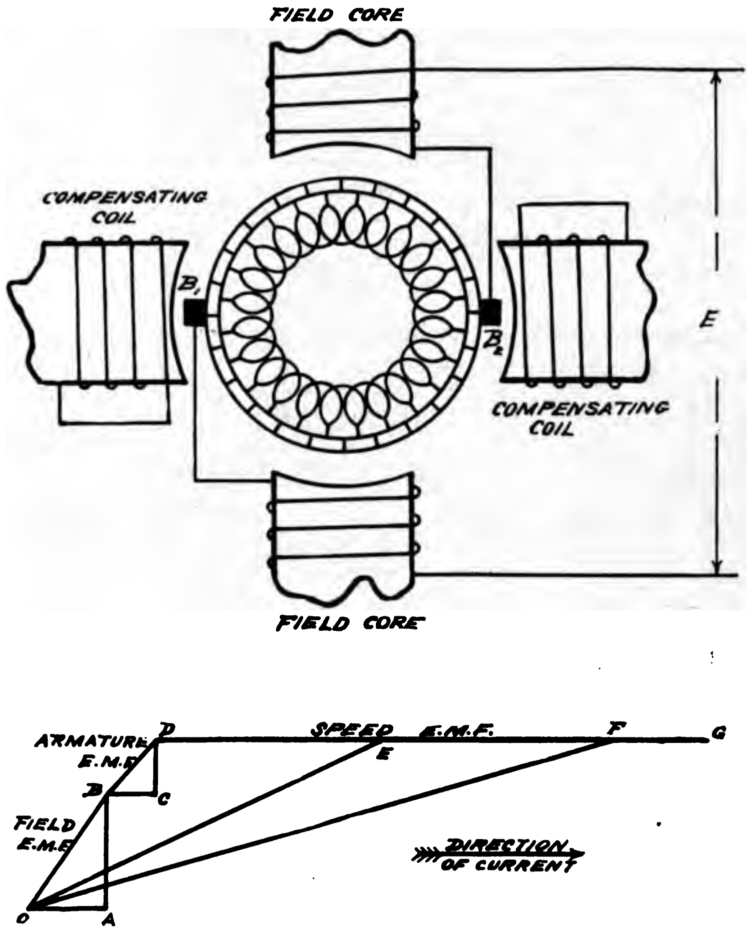


FIG. 10. **Inductively Compensated Series Motor.**

which current flows equal in magneto-motive force and opposite in phase to the current in the armature. This compensating current may be produced inductively by using the stationary winding as the short circuited secondary of a transformer of which the armature is the primary, as illustrated diagrammatically in Fig. 10, or the main line current may be sent di-

rectly through the compensating coil as shown in Fig. 11. In the former case the transformer action is such that the compensation is practically complete, giving minimum combined reactance of the two circuits while in the latter case, the proportion of compensation can be varied at will. It is found that in any case the best general effects are produced when the compensation is complete, and experiments seem to indicate that under such conditions the two methods of compensation differ

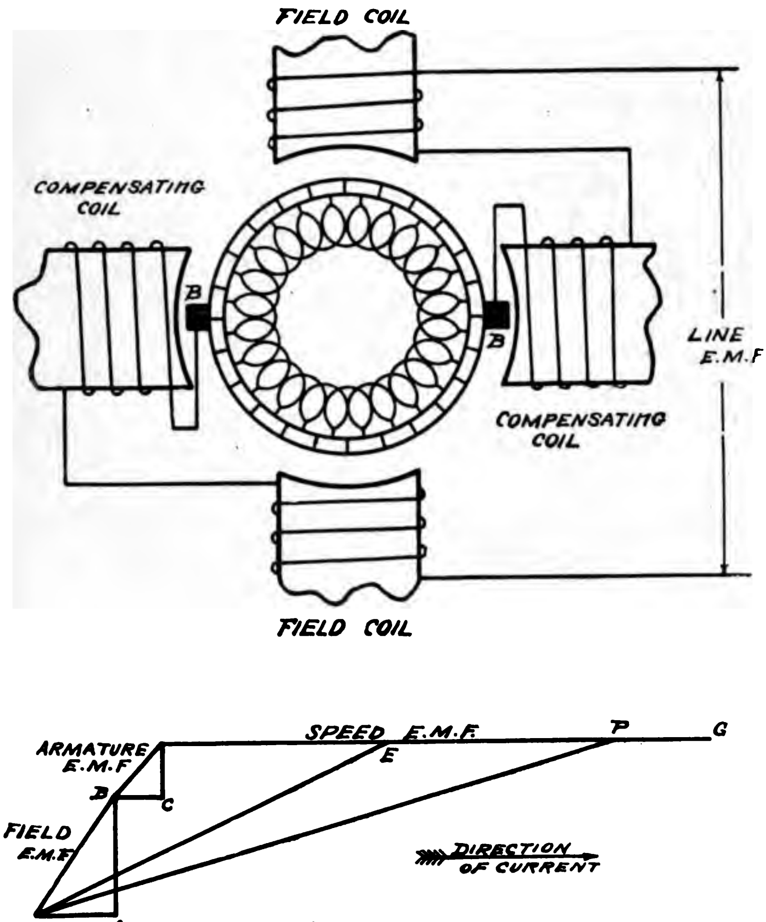


FIG. 11. Conductively Compensated Series Motor.

inappreciably for strictly alternating current work, but that for direct current operation where the forced compensation can

be used to prevent field distortion and improves the commutation, the latter method is preferable.

Referring to Figs. 10 and 11, assume an ideal series motor with complete compensation, letting n be the ratio of effective field to armature turns, at any speed S with synchronism as unity, the apparent impedance of the motor circuits will be

$$Z = X_f \sqrt{\frac{S^2}{n^2} + 1} \tag{154}$$

of which

$$X = X_f \tag{155}$$

represents the reactance of the motor circuits which is confined to the field coil, and of which

$$R = \frac{SX_f}{n} \tag{156}$$

represents the apparent resistance effect of the dyamo speed e.m.f. counter generated at the brushes $B_1 B_2$ due to the cutting of the field flux by the armature conductors, (See eq. 123).

The power factor is

$$\cos\theta = \frac{R}{Z} = \frac{\frac{S}{n}}{\sqrt{\frac{S^2}{n^2} + 1}} \tag{157}$$

which continually approaches positive or negative unity with increase of speed in the corresponding direction.

At synchronism when $S = 1$ the power factor is

$$\cos\theta = \frac{1}{\sqrt{1 + n^2}} \text{ (see eq. 147)} \tag{158}$$

The line current is

$$I = \frac{E}{Z} = \frac{E}{X_f} \cdot \frac{1}{\sqrt{\frac{S^2}{n^2} + 1}} \tag{159}$$

The power is

$$P = EI\cos\theta = \frac{E^2}{X_f} \cdot \frac{\frac{S}{n}}{\frac{S^2}{n^2} + 1} \tag{160}$$

The torque is

$$D=\frac{P}{S}=\frac{E^2}{nX_f}\cdot\frac{1}{\left(\frac{S^2}{n^2}+1\right)}=\frac{I^2X_f}{n} \text{ (see eq. 150)} \qquad (161)$$

The ratio of the torque at synchronous speed to that at standstill is

$$\frac{D_s}{D_o}=\frac{1}{\frac{1}{n^2}+1}=\frac{n^2}{1+n^2} \text{ (see eq. 153)} \qquad (162)$$

which in a practical machine can be made as much smaller than unity as desired by a proper proportioning of the field and armature windings. It is evident, therefore, that such a machine can be made suitable for traction when a proper value of n is chosen.

The above equations refer to ideal motors without resistance and local leakage reactance and devoid of all minor disturbing influences. A close approximation for the effect of the resistance and leakage reactance may be obtained as follows:

Let r_f = resistance of field coil
r_c = resistance of compensating coil (reduced to a 1 to 1, armature ratio)
r_a = resistance of armature
x_f = local reactance of field coil
x_a = combined leakage reactance effect of armature and compensating coils.

Then the apparent impedance is

$$Z=\sqrt{\left(\frac{SX_f}{n}+r_f+r_c+r_a\right)^2+(X_f+x_f+x_a)^2} \qquad (163)$$

Power factor is

$$\cos\theta=\frac{R}{Z}=\frac{\frac{SX_f}{n}+r_f+r_c+r_a}{\sqrt{\left(\frac{SX_f}{n}+r_f+r_c+r_a\right)^2+X_f+x_f+x_a}} \qquad (164)$$

Power input is

$$P=EI\cos\theta=\frac{E^2\left(\frac{SX_f}{n}+r_f+r_c+r_a\right)}{\left(\frac{SX_f}{n}+r_f+r_c+r_a\right)^2+\left(X_f+x_f+x_a\right)^2} \qquad (165)$$

The copper loss and equivalent effective resistance loss will be,

$$I^2R=\frac{E^2(r_f+r_c+r_a)}{\left(\frac{SX_f}{n}+r_f+r_c+r_a\right)^2+(X_f+x_f+x_a)^2} \qquad (166)$$

Electrical output is

$$P - I^2R = \frac{\dfrac{E^2 S X_f}{n}}{\left(\dfrac{SX_f}{n} + r_f + r_c + r_a\right)^2 + \left(X_f + x_f + x_a\right)^2} \qquad (167)$$

The torque is

$$D = \frac{P - I^2 R}{S} = \frac{I^2 X_f}{n} \text{ (see eq. 161)} \qquad (168)$$

The equations here given are represented graphically in the diagrams of Figs. 10 and 11, which show the impedance (e.m.f. for unit current) characteristics of the machines.

$OA = r_f$

$BC = r_a + r_c$

$AB = X_f + x_f$

$DC = X_a$

$DE = \dfrac{S X_f}{n}$ at speed S

$OE = Z$ at speed S

$\cos EOA = \cos \theta =$ power factor at speed S

These characteristics together with the brush short circuiting effect and other minor modifying influences will be discussed in detail in a later paper. It is sufficient here to state that the effect of the short circuit by the brush of a coil in which an active e.m.f. is generated, both by transformer and speed action, tending to increase the apparent impedance effects at high speeds is to some extent balanced by the fact that the flux which causes the generation of a counter e.m.f. by dynamo speed action is out of phase and lagging with respect to the line current and that the counter e.m.f. therefore, tends to lag behind the current or to cause the current to become leading with respect to the counter e.m.f., so that the neglected disturbing influences tend to render the final effect quite small, the result being that the incomplete equations and corresponding graphical diagrams as given above, represent quite closely the observed performance characteristics of the compensated series motors.

ALTERNATING CURRENT COMMUTATOR MOTORS.
IV. INDUCTION-SERIES MOTOR.*

BY A. S. M'ALLISTER.

Excellent performance of the conpensated alternating-current motor may be obtained by using the field coil as the load circuit from the compensating coil employed as the secondary of a transformer, the armature being used as the primary, as diagrammatically represented in Fig. 12. The current which enters the

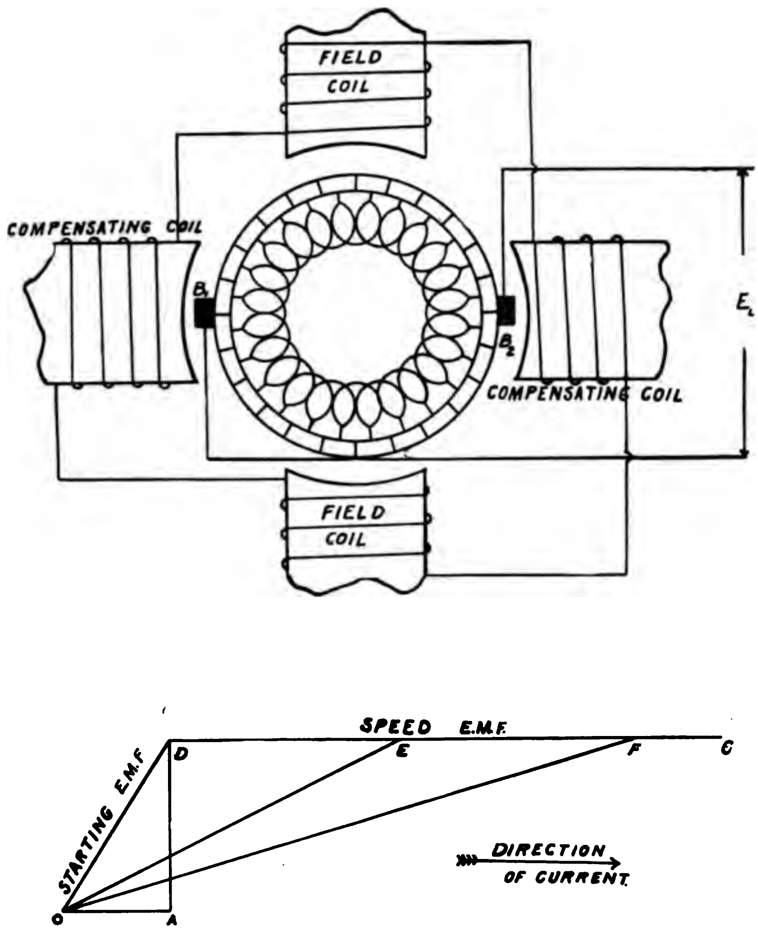


FIG. 12. Induction-Series Motor.

* Thesis for Ph.D. degree, Cornell University.

armature winding through the brushes B_1 B_2 causes the formation on the armature core of magnetic poles having the mechanical direction of the axial line joining the brushes, and the rate of change of the magnetism generates an electromotive force in the compensating coil. Due to this electromotive force, current flows through the locally-closed circuits around the compensating and field coils, and produces magnetic poles in the stationary field-cores.

Consider now the load-circuit surrounding the quadrature field-cores. Since to this winding there is no opposing secondary circuit, the magnetism in the core will be practically in time-phase with the current producing it. This current is the secondary load-current of the transformer. As is true in any transformer, there will flow in the primary coil a current in phase opposition to the secondary current in addition to and superposed upon the primary no-load exciting-current. It is thus seen, that the load-current in the primary (or armature) coil will be in time-phase opposition with the magnetism in the quadrature core. And, since this current and the magnetism reverse signs together, the torque, due to their product and relative mechanical position, will remain always of the same sign—though fluctuating in value. Hence the machine operates similarly to a direct-current series motor.

When the armature revolves at a certain speed, the motion of its conductors through the quadrature magnetic field, generates in the armature winding an electromotive force which appears at the brushes B_1 B_2 as a counter e.m.f. This weakens the effective electromotive force and therewith the armature-current, the armature-core magnetism, the field-current and the field-core magnetism. Thus there results from increased speed of the armature a reduced torque, just as occurs in direct-current series motors. By increasing the applied electromotive force, an increase of torque can be obtained even at excessively high speeds, and the motor tends to increase indefinitely the speed of its armature as the applied electromotive force is increased, or as the counter torque is decreased. There is no tendency to attain a definite limiting speed as is found to be true with revolving field induction-motors and repulsion motors.

Let E_a be the counter transformer e.m.f. across the armature coil, the armature being stationary.

Then

$$E_a = \frac{2\pi f N_a \phi_a}{\sqrt{2}\,10^8}, \qquad \text{see eq. (55)} \qquad (169)$$

where f = frequency in cycles per second
N_a = effective number of armature turns
ϕ_a = maximum value of armature flux cutting the compensating coil

$$N_a = \frac{C}{2\pi} \qquad \text{see eq. (56)} \qquad (170)$$

where C is the actual number of conductors on the armature, a bipolar model being assumed.

$$E_a = \frac{f C \phi_a}{\sqrt{2}\,10^8} \qquad (171)$$

Let N_c = effective number of turns on the compensating coil, then

$$E_c = \frac{2\pi f N_c \phi_a}{\sqrt{2}\,10^8} = \frac{E_a N_c}{N_a} \qquad (172)$$

where E_c = transformer e.m.f. of the compensating coil.
Let N_f = effective number of turns on the field coil then

$$E_f = \frac{2\pi f N_f \phi_f}{\sqrt{2}\,10^8} \qquad (173)$$

where E_f = impressed e.m.f. of the field coil
ϕ_f = maximum value of field flux

$$E_f = E_c, \text{ hence } N_c \phi_a = N_f \phi_f \qquad (174)$$

and

$$\frac{\phi_f}{\phi_a} = \frac{N_c}{N_f} \qquad (175)$$

Let E_v be the e.m.f. counter generated at the brushes $B_1\,B_2$ (Fig. 12) by speed action due to the cutting of the flux ϕ_f by the armature conductors C at speed V revolutions per second, then

$$E_v = \frac{C \phi_f V}{\sqrt{2}\,10^8} \qquad \text{see eq. (59)} \qquad (176)$$

$$V = S f \qquad (177)$$

where S is the speed with synchronism as unity.
Combining (171), (176) and (177)

$$E_v = \frac{S E_a \phi_f}{\phi_a} = \frac{S E_a N_c}{N_f} \qquad (178)$$

Let n be the ratio of effective field to compensating coil turns.

$$E_v = \frac{S E_a}{n} \qquad \text{see eq. (123)} \qquad (179)$$

This electromotive force is in time-phase with the field flux ϕ_f, is in phase opposition with the live current and hence is in time quadrature (leading) with respect to the e.m.f. E_a. The impressed electromotive force E is balanced by the two components, E_v and E_a so that

$$E = \sqrt{E_v^2 + E_a^2} \tag{180}$$

$$E = E_a \sqrt{\frac{S^2}{n^2} + 1} \tag{181}$$

On the basis of unit line current, the electro-motive forces may be treated as impedances, as was done with the repulsion-series and compensated-series motors.

$$Z = X_t \sqrt{\frac{S^2}{n^2} + 1} \tag{182}$$

where $\quad \frac{X_t S}{n} = \underline{R}$ and $X_t = \underline{x}$

X_t being the combined reactance effect of the field, compensating coil and armature circuits.

The power factor is,

$$\cos\theta = \frac{R}{\underline{Z}} = \frac{\frac{S}{n}}{\sqrt{\frac{S^2}{n^2} + 1}} = \frac{S}{\sqrt{S^2 + n^2}} \tag{183}$$

which when $S = 1$ or at synchronism, reduces to

$$\cos\theta = \frac{1}{\sqrt{1 + n^2}} \tag{184}$$

the interpretation of which is that the power factor at synchronism can be caused to approach unity quite closely by the use of a small value of n, that is, by employing a small ratio of field to compensating coil turns. With increase of speed the power factor continually increases for any value of n.

The line current is

$$I = \frac{E}{Z} = \frac{E}{X_t} \frac{1}{\sqrt{\frac{S^2}{n^2} + 1}} = \frac{E\,n}{X_t\sqrt{S^2 + n^2}} \tag{185}$$

The power is

$$P = E\,I\cos\theta = \frac{E^2}{X_t} \cdot \frac{\frac{S}{n}}{\frac{S^2}{n^2} + 1} = \frac{E^2\,S\,n}{X_t\,(S^2 + n^2)} \tag{186}$$

which becomes negative when S reverses, or the machine operates as a generator when driven against its natural tendancy to rotation.

The torque is

$$D = \frac{P}{S} = \frac{E^2 n}{X_t (S^2 + n^2)} = \frac{I^2 X_t}{n} \tag{187}$$

which is maximum at maximum current and retains its sign when S is reversed.

At starting, the torque is

$$D_0 = \frac{E}{X_t} \cdot \frac{n}{n^2} \tag{188}$$

at synchronous speed, the torque is

$$D_s = \frac{E}{X_t} \cdot \frac{n}{(1 + n^2)} \tag{189}$$

and

$$\frac{D_s}{D_0} = \frac{n^2}{1 + n^2} \tag{190}$$

which when $n = 1$ reduces to

$$\frac{D_s}{D_0} = \frac{1}{1 + 1} = .5 \tag{191}$$

and can be given any desired value by a proper selection of n, see eq. (153). A relatively low value of n would produce a machine having the torque characteristics of the direct current series motor and hence one suitable for traction. See eq. (162).

It remains to investigate the relation of the currents in the compensating coil and in the armature circuit (the secondary and primary of the assumed transformer.)

Let i_a be the current which would flow in the armature when the field coil circuit is open. Then i_a is the exciting current of the assumed transformer and it has a value such that its product with the effective number of armature turns, forces the flux, ϕ_a, demanded by the impressed e.m.f., through the reluctance of their paths in the magnetic structure, in line with the brushes B_1 B_2 (Fig. 12). When the field circuit is closed there flows through the field and compensating coil a current i_f, of a value such that its magnetomotive force when flowing through the field turns N_f, produces the flux ϕ_f demanded by the e.m.f. E_f or E_c. The current i_f is in time-phase with the flux ϕ_f and hence is in time quadrature with the e.m.f. E_c, The current i_a is in phase with the flux ϕ_a and in time quadrature with E_a or E_c. When the field circuit is closed a current equal in magnetomotive force

and opposite in phase to i_f is superposed upon i_a in the primary (armature) circuit. These two currents are directly in phase so that the resultant current becomes

$$I = i_a + p\, i_f \qquad (192)$$

where p is a proportionality constant the value of which will be discussed later.

Since both i_a and i_f reach their maximum values simultaneously with ϕ_f, one is led to the highly interesting conclusion that even the exciting current i_a is effective in producing torque by its direct product with the field magnetism, and, that under speed conditions both i_a and $p\, i_f$ are equally effective (per ampere) in producing power.

The relative values of i_a and i_f and of p may be approximated as follows:

Assuming similar conditions for the three coils, the field, the compensating and the armature circuits,—equal reluctance—

$$\frac{i_a N_a}{\phi_a} = \frac{i_f N_f}{\phi_f} \qquad (193)$$

$$N_f = n\, N_c \qquad (194)$$

$$N_c\, \phi_a = N_f\, \phi_f \qquad \text{see eq. (174)} \qquad (195)$$

$$\phi_f = \frac{N_c\, \phi_a}{N_f} \qquad (196)$$

$$i_f = \frac{N_a\, \phi_f}{N_f\, \phi_a} i_a = \frac{N_a N_c}{N_f^2} i_a = \frac{N_a\, i_a}{N_c\, n^2} \qquad (197)$$

From transformer relations there is obtained the equation

$$\frac{N_c}{N_a} = p \qquad \text{see eq. (192)} \qquad (198)$$

Combining (197) and (198)

$$i_f = \frac{i_a}{p\, n^2} \qquad (199)$$

Combining (199) and (192)

$$I = i_a\left(1 + \frac{1}{n^2}\right) \qquad (200)$$

Comparing (199) and (200),

$$i_f = \frac{I \cdot \frac{1}{pn^2}}{1 + \frac{1}{n^2}} = \frac{I}{pn^2 + p} \qquad (201)$$

The relations above expressed depend upon certain assumptions as to the reluctance in line with the armature circuit and

the field coil, and will be modified if the assumptions made are not applicable to the motor as constructed. As a method of reviewing the problem, in a general way, however, the assumption made and the conclusions drawn therefrom are sufficiently exact. In the determination of the equations used above, an ideal motor has been considered, the resistance and local leakage reactance effects being neglected. Actual operating conditions may be more closely represented as follows:

Let

r_f = resistance of field coil.
r_c = resistance of compensating coil.
r_a = resistance of armature.
x_f = local leakage reactance of field coil.
x_c = local leakage reactance of compensating coil.
x_f = local leakage reactance of armature circuit.

Then the copper loss of the motor circuits will be

$$I^2 R_m = I^2 r_a + i_f^2 (r_f + r_c) = I^2 \left[r_a + \frac{r_f + r_c}{(pn^2 + p)^2} \right] \qquad (202)$$

where R_m is the effective equivalent value of the motor-circuit resistance, that is,

$$R_m = r_a + \frac{r_f + r_c}{(pn^2 + p)^2} \qquad (203)$$

Similarly it may be shown that the equivalent effective value of the local leakage reactance of the motor-circuit is

$$X_m = x_a + \frac{x_f + x_c}{(pn^2 + p)^2} \qquad (204)$$

Combining equations (182), (203), and (204), the apparent impedance of the motor-circuits becomes

$$Z = \sqrt{\left[\frac{SX_t}{n} + r_a + \frac{r_f + r_c}{(pn^2 + p)^2} \right]^2 + \left[X_t + x_a + \frac{x_f + x_c}{(pn^2 + p)^2} \right]^2} \qquad (205)$$

The power factor is

$$\cos\theta = \frac{R}{Z} = \frac{\frac{SX_t}{n} + r_a + \frac{r_f + r_c}{(pn^2 + p)^2}}{\sqrt{\left[\frac{SX_t}{n} + r_a + \frac{r_f + r_c}{(pn^2 + p)^2} \right]^2 + \left[X_t + x_a \frac{x_f + x_c}{(pn^2 + p)^2} \right]^2}} \qquad (206)$$

The current is $\frac{E}{Z} = I$. (207)

The input $= EI \cos\theta$. (208)

The output is $P = EI\cos\theta - I^2R_m$. (209)

$$P = \frac{E^2\left[\frac{SX_t}{n} + r_a + \frac{r_f + r_o}{(pn^2+p)^2}\right]}{\left[\frac{SX_t}{n} + r_a + \frac{r_f+r_o}{(pn^2+p)^2}\right]^2 + \left[X_t + x_a + \frac{x_f+x_o}{(pn^2+p)^2}\right]^2} - \tag{210}$$

$$\frac{E^2\left[r_a + \frac{r_f+r_o}{(pn^2+p)^2}\right]}{\left[\frac{SX_t}{n} + r_a + \frac{r_f+r_o}{(pn^2+p)^2}\right]^2 + \left[X_t + x_a + \frac{x_f+x_o}{(pn^2+p)^2}\right]^2}$$

$$P = \frac{E^2SX_t}{Z^2M} = \frac{I^2SX_t}{n} \tag{211}$$

The torque is

$$D = \frac{P}{S} = \frac{I^2X_t}{n} \text{ see eq. (187) and eq. (168)} \tag{212}$$

The graphical diagram of Fig. 12. represents the above impedance equations, (e.m.f. for unit current), where

$$OA = r_a + \frac{r_f + r_o}{(pn^2+p)^2} \tag{213}$$

$$AD = X_t + x_a + \frac{x_f + x_o}{(pn^2+p)^2} \tag{214}$$

$$DF = \frac{SX_t}{n} \text{ at speed } S \tag{215}$$

$$OF = Z \text{ at speed } S \tag{216}$$

$$\cos FOA = \cos\theta = \text{power factor at speed } S \tag{217}$$

Although neglecting certain modifying effects, the graphical diagram represents quite closely the observed performance characteristics of the induction-series motor. An inspection of equation (205) will show that certain values there given may be represented by others of much simplified nature since various terms there contained are constant in any chosen motor.

Let, therefore

$$R = r_a + \frac{r_f + r_o}{(pn^2+p)^2} \tag{218}$$

$$x = X_t + x_a + \frac{x_f + x_o}{(pn^2+p)^2} \tag{219}$$

$$P = \frac{X_t}{n} \tag{220}$$

then the apparent impedance becomes,

$$Z = \sqrt{(R + PS)^2 + X^2} \tag{221}$$

the power factor is,

$$\cos\theta = \frac{R + PS}{\sqrt{(R + PS)^2 + X^2}} \tag{222}$$

which continually approaches unity with increase of speed.

Let rotation of the armature in the direction produced by the electrical (its own) torque be considered positive. Then may rotation in the contrary direction (against its own torque) be considered negative. Since the power component of the motor impedance has a certain value at zero speed, and increases with increase of speed, it should follow that by driving the rotor in a negative direction the apparent power component will reduce to zero and disappear. The power factor then reduces to zero and the current supplied to the motor will represent no energy flowing either to or from the motor.

This will be apparent from the relations above set forth, as well as by the relations algebraically expressed by the equation

$$\text{power} = EI\cos\theta = \frac{E^2(R - PS)}{X^2 + (R - PS)} \tag{223}$$

the negative sign being due to the direction of rotation and the expression reducing to zero for zero value of the apparent power component $R - PS$. A further increase of speed in the negative direction will cause the expression for the power-factor and for the power, to become negative, the interpretation of which is that the machine is now being operated as a generator and hence is supplying energy to the line, that is, energy is flowing from the machine. Fig. 13, which gives the observed performance characteristics of a certain induction-series motor, will serve to show to what extent these theoretical deductions may be realized in an actual machine. If, then, during operation as a motor at a certain speed, the quadrature field flux be relatively reversed with reference to the brush axial-line field flux, so as to tend to drive the armature in the opposite direction, not only will a braking effect be produced by such change but energy will be transmitted from the machine to the line.

The effect of the short circuit by the brush of a coil in which an active e.m.f. is generated, which has been omitted in the above equations, though completely included in the test curves,

may be treated as follows. Referring to Fig. 12 it will be seen that at any speed S there will be generated in the coil under the brush by dynamo speed action an e.m.f.

$$e_s = K \phi_a S \qquad \text{see eq. (43)} \qquad (224)$$

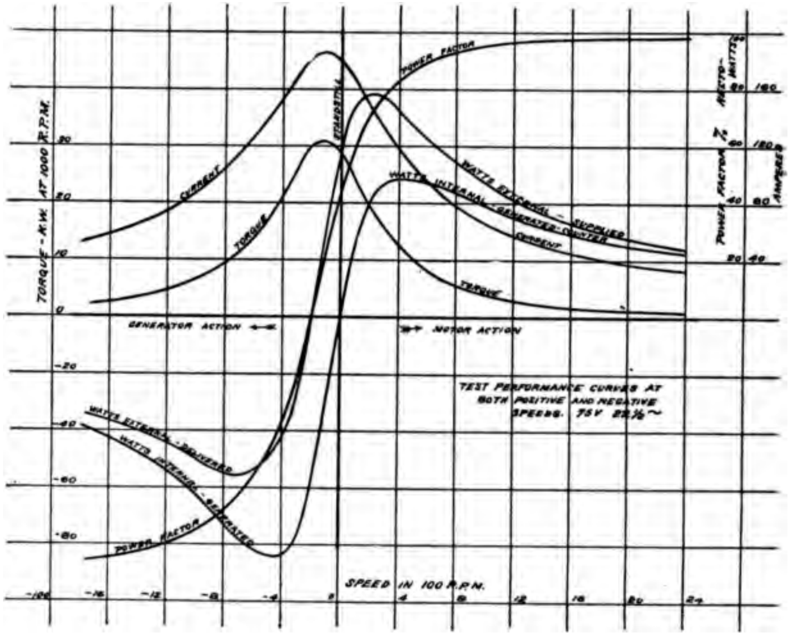


FIG. 13. Test Characteristics of Induction-Series Motor.

where K is constant. This e.m.f. is in time-phase with the flux ϕ_a. In this coil there will also be generated an e.m.f., e_f, by the transformer action of the field flux, such that

$$e_f = K \phi_f \qquad \text{see eq. (44)} \qquad (225)$$

This e.m.f. is in time quadrature to ϕ_f. Since ϕ_f and ϕ_a are in time phase, the component e.m.f.'s acting in the coil under the brush are in time quadrature, so that the resultant e.m.f. is

$$E_b = \sqrt{e_s^2 + e_f^2} = K \sqrt{\phi_a^2 S + \phi_f^2} \qquad (226)$$

$$\frac{\phi}{\phi_f} = n \qquad \text{see eq. (175)} \qquad (227)$$

$$E_b = K \phi_a \sqrt{S^2 + \frac{1}{n^2}} \qquad (228)$$

combining equations (169) and (181)

$$E = \frac{2 \pi f N_a \phi_a}{\sqrt{2} \cdot 10^8} \sqrt{\frac{S^2}{n^2} + 1} \qquad (229)$$

$$\phi_a = \frac{\sqrt{2} \cdot 10^8 \cdot E}{2 \pi f N_a \sqrt{\frac{S^2}{n^2} + 1}} \tag{230}$$

combining (230) and (228)

$$E_b = \frac{K \sqrt{2} \cdot 10^8 E}{2 \pi f N_a} \cdot \frac{\sqrt{S^2 + \frac{1}{n^2}}}{\sqrt{\frac{S^2}{n^2} + 1}} \tag{231}$$

$$E_b = A \sqrt{\frac{S^2 n^2 + 1}{S^2 + n^2}} \tag{232}$$

where A is a constant as found above.

When $n = 1$, E_b is constant, independent of the speed, while when n is very small E_b is large at zero speed and continually decreases with increase of speed. When $S = 1$ or at synchronous speed

$$E_b = \frac{K \sqrt{2}\, 10^8\, E}{2 \pi f N_a} \tag{233}$$

quite independent of the value of n.

The relative impedance effect of E_b can be determined by combining equations (232) and (185) thus

$$\frac{E_b}{I} = \frac{A x_t}{E n} \frac{\sqrt{S^2 n^2 + 1}}{(S^2 + n^2)} \cdot \sqrt{S^2 + n^2} \tag{234}$$

$$\frac{E_b}{I} = B \sqrt{S^2 n^2 + 1} \tag{235}$$

B being a constant. The interpretation of equation (235) is that the apparent impedance effect of the short circuit by the brush, consists of two components in quadrature, one component being of constant value and the other varying directly with the speed. Experimental observations fully confirm these theoretical conclusions, and show that the increase in apparent reactive effect with increase of speed for motor operation is approximately counterbalanced by the lagging counter e.m.f. (leading, current) effect of the time-phase displacement between exciting current and field magnetism as has been mentioned previously and as will be dwelt upon subsequently. During generator operation, that is, with negative value of S, the apparent reactive effect of the short circuit at the brush adds directly to the lagging field flux, counter e.m.f. effect and therefore, the

apparent reactance of the motor circuits increases rapidly with increase of speed in the negative direction, though remaining practically constant for all values of positive speed. These facts will be appreciated from a study of the test characteristics of the induction series machine throughout both its generator and motor operating range as shown in Fig. 13.

Mention has frequently been made of the fact that in the development of the equations for expressing the performance of the various types of series motors the effect of the hysteretic angle of time-phase displacement, between the magnetizing force and the magnetism produced thereby has been neglected. In a *closed* magnet path operated at a density below saturation the tangent of the angle of time-phase displacement will be approximately unity—depending for its exact value upon the quality of the magnetic material. Consider the magnetic and electric circuits of the machine treated as a stationary transformer. The hysteresis loss will be, in watts,

$$W_h = \frac{.0021 f A l B_m^{1.6}}{10^7} \qquad (236)$$

where A = cross sectional area of magnetic path

l = length of magnetic path (in centimeters)

B_m = maximum magnetic density (c.g.s)

The electromotive force counter generated in the transformer coil having N turns will be, in effective volts,

$$E = \frac{2 \pi f A B_m N}{\sqrt{2}\, 10^8} \qquad (237)$$

The current to supply the hysteresis loss will be

$$I_h = \frac{W_h}{E} = \frac{.0021 f A l B_m^{1.6}}{2 \pi f A B_m N} \cdot \frac{\sqrt{2}\, 10^8}{10^7} = .000462\, l B_m^{.6} \qquad (238)$$

With a permeability of μ the magnetizing component of the no-load current will be

$$I_\mu = \frac{A B_m l}{\frac{4\pi}{10} \mu A \sqrt{2} N} = \frac{10}{4\sqrt{2}\pi} \cdot \frac{B_m l}{\mu N} \qquad (239)$$

For a certain value of permeability, depending upon the magnetic density, the hysteresis current and the magnetizing current become equal in value. Thus when the two components of the no-load exciting current become equal $I_\mu = I_h$,

$$\sqrt{2}\, 10 \frac{.0021\, l B_m^{.6}}{2 \pi N} = \frac{B_m \cdot l \cdot 10}{4\pi\sqrt{2} \cdot \mu \cdot N} \qquad (240)$$

from which is obtained,

$$\mu = 119\, B_m{}^{.4} \qquad (250)$$

The meaning of equation (250) is that with a permeability of the value there designated, the hysteresis current and the no-load exciting current are equal in value and that the resultant current $\sqrt{I_h^2 + I_\mu^2}$ is displaced from the flux by a time-phase angle whose tangent (equal at all times to the ratio of I_μ to I_h) is unity, as stated previously. For commercial laminated steel operated at densities below saturation, the permeability differs but slightly from the value given by the equation (250), though with increase of magnetic density above 7,000 lines per square centimeter the permeability falls off rapidly and the tangent of the angle of displacement between flux and current becomes correspondingly increased.

In an open magnetic circuit the permeability of a portion of the path reduces from the value approximately represented by the equation (250) to a value of unity, producing a very marked effect upon the hysteretic angle of displacement between flux and current.

Let l = length of path in magnetic material of permeability μ,
d = length of path in air,

then, assuming that permeability is as represented by equation (250), the tangent of the angle of time-phase displacement between flux and magnetizing force is such that

$$\tan \delta = \frac{\frac{l}{\mu}}{\frac{l}{\mu} + d} = \frac{l}{l + \mu d} \qquad (251)$$

the significance of which equation is that the flux lags behind the current producing it, by an angle which depends for its value largely upon the ratio of the air-gap to the length of the magnetic path. Assigning values to μ, l and d, it will be seen that in any practical case the angle δ must be quite small,—seldom more than 2 degrees.

It should be carefully noted that a slight error is introduced on account of the fact that the permeability of commercial magnetic material undergoes a cyclic change with each alternation of the current, and that, independent of the angle of time-phase displacement between flux and current, the shape of the waves representing the time-values of the two can not both be sinusoidal, and that in assigning a value to the angle of time-phase

displacement between the flux and current, the lack of similarity of the two waves has been neglected.

Under speed conditions the e.m.f. counter generated by the cutting of the armature conductors across the field magnetism, varies in value with the magnetism, and hence it must have a wave shape of time-value similar in all respects to that of the field flux, and must have a time-phase position with reference to the field current quite the same as that of the magnetism. The counter generated speed e.m.f. must, therefore, lag behind the current by an angle whose tangent is as given by equation (251). Now since the counter e.m.f. lags behind the current, the current must lead the counter e.m.f. by the same angle—a fact which has been mentioned previously.

With motors having air-gaps of sizes demanded by mechanical clearance, the inherent angle of lead is quite small, and its effect upon the power-factor is neutralized by the effect of the short circuit by the brush of a coil in which is generated an e.m.f. by both transformer and speed action when the machine is operated as a motor. When the machine is operated as a generator, however, the hysteretic angle and the angle due to the short circuiting effect are in a direction such as to be additive to the stationary reactive effect of the motor circuits and, therefore, during generator operation the power factor is lower than during motor operation as shown in Fig. 13.

While the angle of lead due to the hysteretic effect, even when the machine is running as a motor, is in any case quite small and its good effects cannot be availed of, it is possible by means of certain auxiliary circuits to give to the angle of time-phase displacement between the line current and the flux any value desired, and thus to cause the operating power factor to become unity or to decrease with leading wattless current, as is shown below.

Fig. 14 represents diagrammatically the circuits of a conductively compensated-series motor in parallel with the field coil of which is placed a non-inductive resistance. Consider first, ideal conditions in which the armature and compensating coils are without resistance and the compensation is complete so that these two circuits, treated as one, are without inductance. The field coil is without resistance but constitutes the reactive portion of the motor circuits.

When the armature is stationary the circuit through the resis-

tance being open, the current taken by the machine has a value determined by the ratio of the impressed e.m.f. and the reactance of the field coil. This current lags 90 time degrees behind the e.m.f. across the field coils. When a resistance is placed in shunt to the field coil, current flows therethrough, quite independently of the field current. The current taken by the resistance is in time-phase with the e.m.f. impressed upon the field coil.

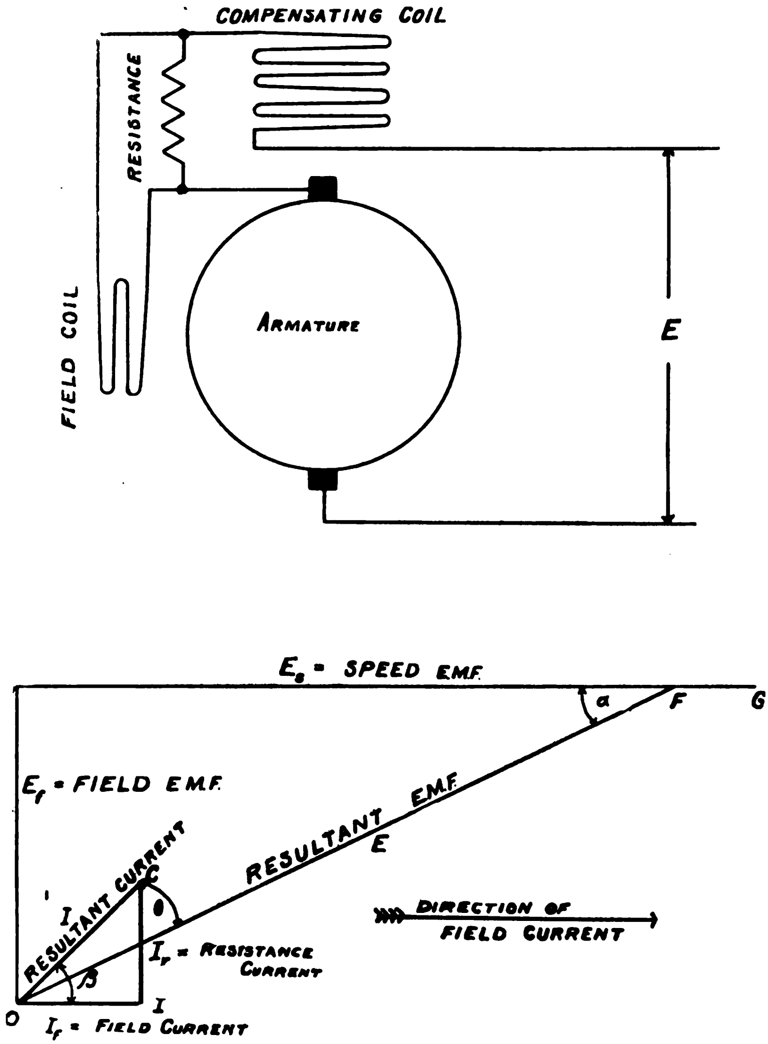


FIG. 14.—Compensated Series Motor with Shunted Field Coil.

In Fig. 14 let $OI = I_f$ represent the field current, assumed always of unit value. $OD = E_f$ is the e. m. f. impressed across the field coil and the shunted resistance. I_r is the current taken by the resistance. $OC = I$, the current which flows through the armature and compensating coil or the resultant current taken by the motor has a value represented by the equation

$$I = \sqrt{I_f^2 + I_r^2} \qquad (252)$$

and has a phase displacement β with reference to the field current such that

$$\tan \beta = \frac{I_r}{I_f} \qquad (253)$$

With unit value of field current, under speed conditions, the e.m.f., E_s, (DF of Fig. 14) counter generated at the brushes, due to the presence of the field flux, will be proportional directly to the speed and in time-phase with the field current. Thus this component of the counter e.m.f. of the motor is in no wise affected by the presence of the current through the shunted resistance. At a certain speed, the counter generated armature e.m.f. will have a value represented by the line DF Fig. 14 the resultant e.m.f. $E = OF$ being the vector (quadrature) sum of the speed e.m.f. and the stationary e.m.f. E_s that is

$$E = \sqrt{E_f^2 + E_s^2} \qquad (254)$$

and has a time-phase α position with reference to the speed e.m.f. E_s such that

$$\tan \alpha = \frac{E_f}{E_s} \qquad (255)$$

An inspection of Fig. 14 will show that under operating conditions, the angle of time-phase displacement between the current and the electromotive force, θ, has a value represented by the equation

$$\theta = \beta - \alpha \qquad (256)$$

or the current *leads* the e.m.f. by the angle θ. At a certain critical speed for each value of shunted resistance, or at a certain value of resistance for any given speed, the angle θ reduces to zero, and the power factor of the motor becomes unity.

It is interesting to observe the effect of removing the resistance from in shunt with the field circuit. Since the current taken by the resistance is 90 time-degrees from the field flux, the resultant torque due to the product of this component of the current and the flux is of zero value, the instantaneous torque alternat-

ing at double the circuit frequency. The current through the resistance, therefore, contributes in no way to the power of the machine or to the counter-generated, armature-speed e.m.f., and when the circuit through the resistance is opened no effect whatsoever is produced upon the value of the current taken by the field coil, the counter e.m.f. or the torque of the machine. It is apparent, therefore, that the use of the shunted resistance increases the circuit current in a certain definite proportion, the added component being a leading "wattless" current under speed conditions. If a reactance be placed in parallel with the field coil, the current which flows therethrough will be in time-phase with the field flux, and the torque produced thereby will add to the torque due to the field current and it will affect directly the whole performance of the machine. The current taken by a condensance in shunt with the field coil will be in time-phase opposition to the field current and will tend to decrease directly both the circuit current and the armature torque. An excess of condensance will cause the torque to reverse and the machine to act as a generator even when the speed is in a positive direction. When the condensance and the field reactance are just equal, the circuit current reduces to zero and the torque disappears. Under the conditions here assumed, the counter generated e.m.f. at the armature remains proportional to the product of the field flux and the speed, and there appears the remarkable combination of zero current being transmitted over a certain counter e.m.f. (that is, through infinite impedance) to divide into definite active currents at the end of the transmission circuits.

From what has been demonstrated above, it is seen that shunted condensance acts to take current in phase opposition and to decrease the torque; reactance takes current directly in phase, and increases the torque, while resistance takes current in leading quadratures with the field current and has no effect upon the torque. It is evident that the improvement in power factor due to the use of the resistance is advantageous provided the losses caused by the resistance are not excessive. Referring to Fig. 14, when the resistance is not used the power taken by the machine under speed conditions is

$$P = OI \cdot OF \cdot \cos FOI = I_f E \cos \alpha = I_f E_s \qquad (257)$$

When the machine is stationary, the power absorbed by the resistance is

$$P_r = CI \cdot OD = I_r E_f \qquad (258)$$

When the motor is running with shunted field coil, the power delivered to the machine is

$$P_t = OC \cdot OF \cdot \cos COF = IE \cos \theta \tag{259}$$

$$\theta = \beta - \alpha \tag{250}$$

$$\cos \theta = \cos \beta \cos \alpha + \sin \beta \sin \alpha \tag{261}$$

$$P_t = I \cos \beta \cdot E \cos \alpha + I \sin \beta \cdot E \sin \alpha \tag{262}$$

$$P_t = I_f E_s + I_r E_f = P + P_r \tag{263}$$

The significance of equation (263) is that the energy absorbed is that incident to the use of the resistance, and that for a given current it is unaffected by the speed e.m.f. Thus the current taken by the resistance multiplies into the stationary transformer e.m.f. to give the actual watts absorbed while the same current multiplies into the speed e.m.f. to give apparent leading wattless power.

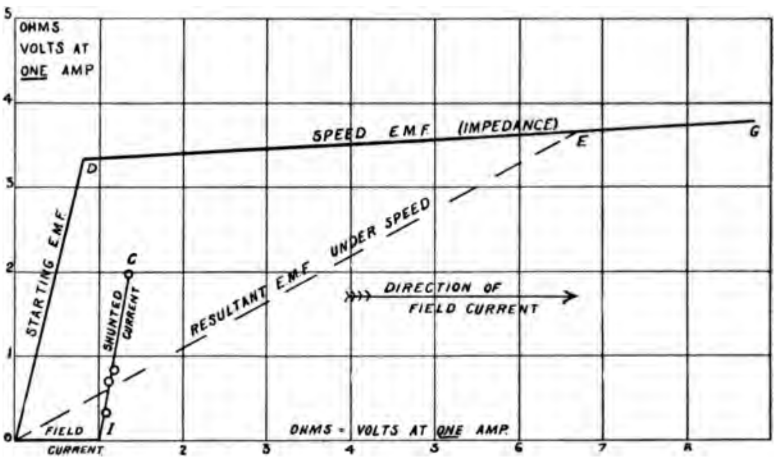


FIG. 15. Observed E.M.F.—Current Characteristics of Plain Series Motor with Shunted Field.

In the derivation of the above equations ideal conditions have been assumed, which cannot be obtained in a practical motor. motor. Fig. 15 represents the observed e.m.f. current characteristics of a certain plain, uniform reluctance motor (see Fig. 9) with shunted field coils, and serves to show that even such an unfavorable machine may be caused to operate at unity power factor at any speed greater than about one-half synchronism.